Solvability of L^p Poisson–Robin(-Regularity) Problems on Rough Domains

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Outline

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2. **Solvability of the L^p Poisson–Neumann problem**

3. **Solvability of the L^p Poisson–Robin problem**

1. Solvability of the L^p Poisson–Dirichlet problem

L^p Dirichlet Problem/ (1)

▸ Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded one-sided chord-arc domain (CAD), $L := -\mathrm{div}(A\nabla\cdot)$ with A satisfying uniform elliptic condition.

- chord-arc domain (CAD): (interior and exterior) corkscrew conditions (CS) + Harnack chain condition (HC) + $\partial\Omega$ is $(n-1)$-Ahlfors regular (AR)
- one-sided CAD: interior corkscrew condition + Harnack chain condition (HC) + $\partial\Omega$ is $(n-1)$-Ahlfors regular (AR)

L^p Dirichlet Problem/ (2)

▸ Let $p \in (1,\infty)$. The Dirichlet problem (D_p^L) is said to be solvable if $\exists\,!\ u$ satisfies

$$(\mathrm{D}_p^L) \quad \begin{cases} -\mathrm{div}(A\nabla u) = 0 & \text{in } \Omega \\ u = g \in L^p(\partial\Omega) & \text{on } \partial\Omega \\ \widetilde{\mathcal{N}}(u) \in L^p(\partial\Omega) \end{cases}$$

and $\|\widetilde{\mathcal{N}}(u)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}$.

• The non-tangential maximal function of u:

$$\widetilde{\mathcal{N}}(u)(\xi) := \sup_{x\in\gamma(\xi)} \left(\fint_{B(x,\frac{1}{4}\delta(x))} |u(z)|^2\, dz \right)^{1/2} \text{ for } \xi \in \partial\Omega,$$

$$\gamma(\xi) := \{x \in \Omega : |x-\xi| < 2\delta(x)\}.$$

L^p Dirichlet Problem/ (3)

- If $L = -\Delta$ and Ω C^1 domain, then (D^L_p) is solvable for $\forall\ p \in (1, \infty]$
  [Fabes, Jodeit & Riviére, Acta Math. (1978)]
- If $L = -\Delta$ and Ω Lipschitz domain, then (D^L_p) is solvable for $p \in (2 - \varepsilon, \infty]$
  [Dahlberg, ARMA (1977); Verchota, J. Funct. Anal. (1984)]
- If $L = -\Delta$ and Ω CAD, then (D^L_p) is solvable for $p \in (p_0, \infty]$ with $p_0 \in (1, \infty)$
  [David & Jerison, Indiana Univ. Math. J. (1990); Semmes, Trans. Amer. Math. Soc. (1989)]

▸ (D^L_p) solvable $\Longleftrightarrow$ $\omega^x \in RH_{p'}(\partial\Omega)$ (harmonic measure)

L^p Dirichlet Problem/ (4)

- $L := -\mathrm{div}(A\nabla\cdot)$ with A smooth, Ω Lipschitz domain, (D_p^L) solvable for $p \in (2-\varepsilon, \infty]$
[Jerison & Kenig, Ann. Math. (1981)]
- If $L := -\mathrm{div}(A\nabla\cdot)$ with A satisfying DKP condition, Ω satisfies CS + AR + IBPCAD
(**IBPCAD**: interior big pieces of chord-arc domains)
$\Rightarrow \exists\ p > 1$ s.t. $(\mathrm{D}_{p'}^L)$ is solvable (see [AHMMT20])
- Particularly, if Ω satisfies CS + AR + IBPCAD, then $\exists\ p > 1$ s. t. $(\mathrm{D}_{p'}^{-\Delta})$ is solvable

- [AHMMT20] J. Azzam, S. Hofmann, J. M. Martell, M. Mourgoglou & X. Tolsa, Harmonic measure and quantitative connectivity: geometric characterization of the L^p-solvability of the Dirichlet problem, Invent. Math. **222** (2020), 881-993.

L^p Poisson–Dirichlet(-Regularity) Problem/ (1)

- Mourgoglou et al. [MPT25] introduced the solvability of the L^p Poisson–Dirichlet(-Regularity) problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = h - \mathrm{div}\boldsymbol{F} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

  - Poisson–Dirichlet problem $(\mathrm{PD}_{p'}^{L})$ is solvable if $\|\widetilde{\mathcal{N}}(u)\|_{L^{p'}(\partial\Omega)} \lesssim \|\widetilde{\mathcal{C}}(\delta^2 h)\|_{L^{p'}(\partial\Omega)} + \|\widetilde{\mathcal{C}}(\delta \boldsymbol{F})\|_{L^{p'}(\partial\Omega)}$.
  - Poisson–Dirichlet-Regularity problem (PDR_{p}^{L}) is solvable if $\|\widetilde{\mathcal{N}}(\nabla u)\|_{L^{p}(\partial\Omega)} \lesssim \|\widetilde{\mathcal{C}}(\delta h)\|_{L^{p}(\partial\Omega)} + \|\widetilde{\mathcal{C}}(\boldsymbol{F})\|_{L^{p}(\partial\Omega)}$.

- The Carleson functional: for $\xi \in \partial\Omega$,

$$\widetilde{\mathcal{C}}(h)(\xi) := \sup_{r>0} \frac{1}{r^{n-1}} \int_{B(\xi,r)\cap\Omega} \left(\fint_{B(y,\frac{1}{4}\delta(y))} |h|^2 \, dz \right)^{1/2} \frac{dy}{\delta(y)}.$$

L^p Poisson–Dirichlet(-Regularity) Problem/ (2)

- Mourgoglou et al. [MPT25] proved the following assertions are equivalent.
  - $(\mathrm{D}^{L}_{p'})$ is solvable $\Leftrightarrow$ $(\mathrm{PD}^{L}_{p'})$ is solvable $\Leftrightarrow$ $(\mathrm{PDR}^{L^*}_{p})$ is solvable
- Key tool: Duality between $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{C}}$ [Hytönen & Rosén, Ark. Math. (2013)]
  - $\left|\int_\Omega uh\,dx\right| \lesssim \|\widetilde{\mathcal{N}}(u)\|_{L^p(\partial\Omega)}\|\widetilde{\mathcal{C}}(\delta h)\|_{L^{p'}(\partial\Omega)}$
  - $\|\widetilde{\mathcal{N}}(u)\|_{L^p(\partial\Omega)} \lesssim \sup_{h:\|\widetilde{\mathcal{C}}(\delta h)\|_{L^{p'}(\partial\Omega)}=1} \left|\int_\Omega uh\,dx\right|$

- [MPT25] M. Mourgoglou, B. Poggi & X. Tolsa, Solvability of the Poisson–Dirichlet problem with interior data in $L^{p'}$-carleson spaces and its applications to the L^p-regularity problem, J. Eur. Math. Soc. (JEMS) (2025), doi: 10.4171/JEMS/1660.

L^p Poisson–Dirichlet(-Regularity) Problem/ (3)

- Initial motivation & application of Mourgoglou et al. [MPT25]: Clarify the relationship between $(\mathrm{D}_{p'}^{L^*})$ and (Re_p^L) (A question raised by Kenig [CBMS Regional Conference Series in Mathematics 83, 1994]).
- The regularity problem (Re_p^L) is solvable if $\exists\,!\ u$ satisfies

$$(\mathrm{Re}_p^L) \qquad \begin{cases} -\mathrm{div}(A\nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

and

$$\|\widetilde{\mathcal{N}}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{\dot{W}^{1,p}(\partial\Omega)},$$

where $\|g\|_{\dot{W}^{1,p}(\partial\Omega)} := \inf_{f\in\mathcal{D}(g)} \|f\|_{L^p(\partial\Omega)}$ and
$\mathcal{D}(g) := \{|g(x) - g(y)| \le |x-y|\,(f(x) + f(y))$ for σ-a. e. $x, y \in \partial\Omega\}$ ([MT24]).

- [MT24] M. Mourgoglou and X. Tolsa, The regularity problem for the Laplace equation in rough domains, Duke Math. J. **173** (2024), 1731-1837.

L^p Poisson–Dirichlet(-Regularity) Problem/ (4)

- Ω Lipschitz domain, $\forall\, p \in (1,\infty)$, $(\mathrm{D}_{p'}^{-\Delta}) \Leftrightarrow (\mathrm{Re}_p^{-\Delta})$ [Verchota, J. Funct. Anal. (1984)]
- Ω Lipschitz domain, $\forall\, p \in (1,\infty)$, $(\mathrm{Re}_p^{L}) \Rightarrow (\mathrm{D}_{p'}^{L^*})$ [Kenig & Pipher, Invent. Math. (1993)]
- Ω CAD, $\forall\, p \in (1,\infty)$, $(\mathrm{Re}_p^{L}) \Rightarrow (\mathrm{D}_{p'}^{L^*})$ [Mourgoglou & Tolsa, Duke Math. J. (2024)]
- Ω CAD, $\forall\, p \in (1,\infty)$, $(\mathrm{D}_{p'}^{-\Delta}) \Leftrightarrow (\mathrm{Re}_p^{-\Delta})$ [Mourgoglou & Tolsa, Duke Math. J. (2024)]
- Ω CAD, L satisfies DKP condition (sharp), $\forall\, p \in (1,\infty)$, $(\mathrm{D}_{p'}^{L}) \Leftrightarrow (\mathrm{Re}_p^{L^*})$ [Mourgoglou, Poggi & Tolsa, J. Eur. Math. Soc. (2025)]

2. Solvability of the L^p Poisson–Neumann problem

L^p Neumann Problem/ (1)

- Let $p \in (1, \infty)$. The Neumann problem (N_p^L) is said to be solvable if $\exists\,!\ u$ up to constants satisfies

$$(\mathrm{N}_p^L) \quad \begin{cases} -\mathrm{div}(A\nabla u) = 0 & \text{in } \Omega \\ A\nabla u \cdot \boldsymbol{\nu} = g \in L^p(\partial\Omega) & \text{on } \partial\Omega \\ \widetilde{\mathcal{N}}(\nabla u) \in L^p(\partial\Omega) \end{cases}$$

and

$$\|\widetilde{\mathcal{N}}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}.$$

L^p Neumann Problem/ (2)

- If $L = -\Delta$, Ω C^1 domain, then (N_p^L) is solvable for $p \in (1, \infty)$ [Fabes, Jodeit & Riviére, Acta Math. (1978)]
- If $L = -\Delta$, Ω Lipschitz domain, then (N_p^L) is solvable for $p \in (1, 2+\varepsilon)$ [Jerison & Kenig, Bull. Amer. Math. Soc. (1981), $p = 2$; Dahlberg & Kenig, Ann. Math. (1987), $p \in (1, 2+\varepsilon)$]
- If $L = -\Delta$, Ω (semi-)convex domain, then (N_p^L) is solvable for $p \in (1, \infty)$ [Geng & Shen, J. Funct. Anal. (2010); Y., Nonlinear Anal. (2016)]
- If $L = -\Delta$, Ω CAD, $n = 2$, $(\mathrm{D}_{p'}^{-\Delta}) \Leftrightarrow (\mathrm{Re}_p^{-\Delta}) \Leftrightarrow (\mathrm{N}_p^{-\Delta})$ [Jerison & Kenig, Math. Scand. (1982)]
- If Ω CAD, $n \geq 3$, even when $L = -\Delta$, the solvability of (N_p^L) is open [Kenig, CBMS Regional Conference Series in Mathematics 83, 1994, Problem 3.2.2]

L^p Neumann Problem/ (3)

▸ The isolation of the L^p Neumann problem:
When $n \geq 3$, there are counterexamples for

- $(\mathrm{D}_{p'}^{L^*}) \not\Rightarrow (\mathrm{N}_p^L)$, $(\mathrm{N}_p^L) \not\Rightarrow (\mathrm{D}_{p'}^{L^*})$, $(\mathrm{Re}_p^L) \not\Rightarrow (\mathrm{N}_p^L)$, $(\mathrm{N}_p^L) \not\Rightarrow (\mathrm{Re}_p^L)$ (see [KP95])

▸ Extrapolation result: Ω starlike Lipschitz domain (see [KP93])

$$(\mathrm{Re}_p^L) + (\mathrm{N}_p^L) \Rightarrow (\mathrm{N}_q^L) \quad \text{for } q \in (1, p + \varepsilon)$$

- [KP95] C. E. Kenig & J. Pipher, The Neumann problem for elliptic equations with nonsmooth coefficients. II, Duke Math. J. **81** (1995), 227-250 (1996).
- [KP93] C. E. Kenig and J. Pipher, The Neumann problem for elliptic equations with nonsmooth coefficients, Invent. Math. **113** (1993), 447-509.

L^p Poisson–Neumann(-Regularity) Problem/ (1)

► Feneuil & Li [arXiv: 2406.16735] introduced the solvability of the L^p Poisson–Neumann(-Regularity) problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = h - \mathrm{div}\boldsymbol{F} & \text{in } \Omega \\ A\nabla u \cdot \boldsymbol{\nu} = \boldsymbol{F} \cdot \boldsymbol{\nu} & \text{on } \partial\Omega \end{cases}$$

- Let $h \equiv 0$. Poisson–Neumann problem $(\mathrm{PN}^L_{p'})$ is solvable if $\int_\Omega u\,dx = 0$ and $\|\widetilde{\mathcal{N}}(u)\|_{L^{p'}(\partial\Omega)} \lesssim \|\widetilde{\mathcal{C}}(\delta \boldsymbol{F})\|_{L^{p'}(\partial\Omega)}$
- Poisson–Neumann-Regularity problem (PNR^L_p) is solvable if $\int_\Omega u\,dx = 0$ and $\|\widetilde{\mathcal{N}}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\widetilde{\mathcal{C}}(\delta|h|)\|_{L^p(\partial\Omega)} + \|\widetilde{\mathcal{C}}(\boldsymbol{F})\|_{L^p(\partial\Omega)}$

L^p Poisson–Neumann(-Regularity) Problem/ (2)

- Feneuil & Li [arXiv: 2406.16735] proved:
  - $(\mathrm{PN}_{p'}^{L^*}) \Longleftrightarrow (\mathrm{PNR}_p^L)$
  - $(\mathrm{PN}_{p'}^{L^*}) \Longleftrightarrow (\mathrm{PNR}_p^L) \Longrightarrow (\mathrm{N}_p^L) \xrightarrow{+(\mathrm{D}_{p'}^{L^*})} (\mathrm{PNR}_p^L)$
  - Application: $(\mathrm{D}_{p'}^{L^*}) + (\mathrm{N}_p^L) \Longrightarrow (\mathrm{N}_q^L)$ for $q \in (1, p+\varepsilon)$ (the extrapolation property of the solvability of L^p Neumann problem) [improves the extrapolation theorem obtained by Kenig & Pipher, Invent. Math. (1993)]

3. Solvability of the L^p Poisson–Robin problem

L^p Robin Problem/ (1)

▶ Let Ω be a bounded one-sided CAD, L a uniformly elliptic operator, $0 \le \alpha \in L^{n-1+\epsilon_0}(\partial\Omega)$, $\alpha \ge c_0$ on $E_0 \subset \partial\Omega$, $\epsilon_0, c_0 \in (0,\infty)$, $\sigma(E_0) > 0$

▶ Let $p \in (1,\infty)$. The Robin problem (R_p^L) is said to be solvable if $\exists\,!\; u$ satisfies

$$(\mathrm{R}_p^L) \quad \begin{cases} -\mathrm{div}(A\nabla u) = 0 & \text{in } \Omega \\ A\nabla u \cdot \nu + \alpha u = g \in L^p(\partial\Omega) & \text{on } \partial\Omega \\ \widetilde{\mathcal{N}}(\nabla u) \in L^p(\partial\Omega) \end{cases}$$

and

$$\|\widetilde{\mathcal{N}}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|g\|_{L^p(\partial\Omega)}.$$

L^p Robin Problem/ (2)

- If $L = -\Delta$, Ω Lipschitz domain, (R_p^L) is solvable for $p \in (1, 2+\varepsilon)$
  [Lanzani & Shen, Comm. Partial Differential Equations (2004)]
- If $L = -\Delta$, Ω C^1 domain, (R_p^L) is solvable for $p \in (1, \infty)$ (when $p > n$, $\alpha \in L^p(\partial\Omega)$)
  [Lanzani & Shen, Comm. Partial Differential Equations (2004)]
- If $L = -\Delta$, Ω semi-convex domain, (R_p^L) is solvable for $p \in (1, \infty)$
  [Y., Yang & Yuan, J. Differential Equations (2018)]
- If Ω CAD, even when $L = -\Delta$, the solvability of (R_p^L) is open.

L^p Poisson–Robin(-Regularity) Problem/ (1)

▶ We introduced the solvability of the L^p Poisson–Robin(-Regularity) problem

$$\begin{cases} -\mathrm{div}(A\nabla u) = h - \mathrm{div}\boldsymbol{F} & \text{in } \Omega \\ A\nabla u \cdot \boldsymbol{\nu} + \alpha u = \boldsymbol{F} \cdot \boldsymbol{\nu} & \text{on } \partial\Omega \end{cases}$$

- Let $h \equiv 0$. Poisson–Robin problem $(\mathrm{PR}^L_{p'})$ is solvable if

$$\|\widetilde{\mathcal{N}}(u)\|_{L^{p'}(\partial\Omega)} \lesssim \|\widetilde{C}(\delta \boldsymbol{F})\|_{L^{p'}(\partial\Omega)}$$

- Poisson–Robin-Regularity problem (PRR^L_p) is solvable if

$$\|\widetilde{\mathcal{N}}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\widetilde{C}(\delta|h|)\|_{L^p(\partial\Omega)} + \|\widetilde{C}(\boldsymbol{F})\|_{L^p(\partial\Omega)}$$

L^p Poisson–Robin(-Regularity) Problem/ (2)

- Theorem (Fu, Yang & Y., 2025). Let $p \in (1, \infty)$.
  - $(\mathrm{PR}_{p'}^{L^*}) \Longleftrightarrow (\mathrm{PRR}_p^L)$
  - $(\mathrm{PR}_{p'}^{L^*}) \Longleftrightarrow (\mathrm{PRR}_p^L) \Longrightarrow (\mathrm{R}_p^L) \overset{+(\mathrm{D}_{p'}^{L^*})}{\Longrightarrow} (\mathrm{PRR}_p^L)$
  - Application: $(\mathrm{D}_{p'}^{L^*}) + (\mathrm{R}_p^L) \Longrightarrow (\mathrm{R}_q^L)$ for $q \in (1, p+\varepsilon)$ (the extrapolation property of the solvability of L^p Robin problem)

- Corollary. Let Ω be a bounded Lipschitz domain, $L := -\Delta$. Then $(\mathrm{PR}_p^{-\Delta})$ is solvable for any $p \in (2-\varepsilon, \infty)$, $(\mathrm{PRR}_p^{-\Delta})$ for any $p \in (1, 2+\varepsilon)$.

L^p Poisson–Robin(-Regularity) Problem/ (3)

- Main ingredients for proofs:
- Duality between $\widetilde{\mathcal{N}}$ and $\widetilde{C}$
- The trace theorem in Sobolev spaces for one-sided CAD
- $$\begin{cases} -\mathrm{div}(A\nabla u) = 0 & \text{in } \Omega \\ A\nabla u \cdot \boldsymbol{\nu} + \alpha u = f & \text{on } \partial\Omega. \end{cases}$$

$$(\mathrm{R}_p^L) \quad \|\widetilde{\mathcal{N}}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

$$(\widetilde{\mathrm{R}}_p^L) \quad \|\widetilde{\mathcal{N}}(\nabla u)\|_{L^p(\partial\Omega)} + \|\widetilde{\mathcal{N}}(u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

Proposition. $(\mathrm{R}_p^L) \Leftrightarrow (\widetilde{\mathrm{R}}_p^L)$.

- Meyers type estimate for local homogeneous Robin problem:

$$\left(\fint_{B\cap\Omega} [|\nabla u| + |u|]^{2+\varepsilon}\, dx\right)^{\frac{1}{2+\varepsilon}} \lesssim \left(\fint_{4KB\cap\Omega} |\nabla u| + |u|\, dx\right).$$

Thank you for your attention.