Capacitary Muckenhoupt Weights and Weighted Norm Inequalities for Hardy-Littlewood Maximal Operators

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Overview

1. Background and classical results

Background

The classical Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by setting

$$\mathcal{M}(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy, \quad \forall\, x \in \mathbb{R}^n \text{ and } f \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Let $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w \in (0, \infty)$ almost everywhere. Given $p \in [1, \infty)$ and a measurable set E, let $w(E) := \int_E w(x)\, dx$ and let f belong to weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$, i.e.,

$$\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx < \infty.$$

Background

As one of fundamental results in harmonic analysis, it is well known that:

- If $p \in (1, \infty)$, then the strong-type weighted norm inequality

$$\int_{\mathbb{R}^n} [\mathcal{M}(f)(x)]^p \, w(x) \, dx \lesssim \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx, \tag{1.1}$$

holds true iff w belongs to the Muckenhoupt class A_p, i.e.

$$[w]_{A_p} := \sup_Q \left[\frac{1}{|Q|} \int_Q w(x) \, dx \right] \left[\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} \, dx \right]^{p-1} < \infty$$

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with the supremum taken over all cubes Q;

- If $p = 1$, then the weak-type weighted norm inequality

$$w(\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > t\}) \lesssim \frac{1}{t} \int_{\mathbb{R}^n} |f(x)| w(x) \, dx, \quad \forall \, t \in (0, \infty), \tag{1.2}$$

holds true iff w belongs the Muckenhoupt class A_1, i.e.

$$w : \ [w]_{A_1} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \frac{\mathcal{M} w(x)}{w(x)} < \infty.$$

Background

- The original proof was given by Muckenhoupt, TAMS, 1972, based on an interpolation argument and a self-improving property of A_p weights:

$$w \in A_p \quad \text{implies} \quad w \in A_{p-\varepsilon} \quad \text{for some} \quad \varepsilon > 0.$$

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- Later, Coifman-Fefferman, Studia Math., 1974, gave a simplified proof by a crucial reverse Hölder inequality: for any $w \in A_p$ with $p \in [1, \infty)$, there exists a $\gamma > 0$ such that, for any cube $Q \subset \mathbb{R}^n$

$$\left[\frac{1}{|Q|}\int_Q w(x)^{1+\gamma}\, dx\right]^{\frac{1}{1+\gamma}} \lesssim \frac{1}{|Q|}\int_Q w(x)\, dx. \tag{1.3}$$

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- After that, an elementary proof for (1.1) avoiding the reverse Hölder inequality was provided by Christ-Fefferman, PAMS, 1983, based essentially on the Calderón-Zygmund decomposition.

Given a set $E \subset \mathbb{R}^n$ and $\delta \in (0, n]$, the Hausdorff content $\mathcal{H}^\delta_\infty(E)$ of E of dimension δ is defined by setting

$$\mathcal{H}^\delta_\infty(E) := \inf \left\{ \sum_i |Q_i|^{\delta/n} : \; E \subset \bigcup_i Q_i \right\},$$

where the infimum is taken over all finite or countable cubes coverings $\{Q_i\}_i$ of E.

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- If $\delta = n$, the Hausdorff content $\mathcal{H}^n_\infty$ is equivalent to the Lebesgue measure on $\mathbb{R}^n$.

For $p \in (0, \infty)$ and a function g on $\mathbb{R}^n$, its Choquet integral with respect to Hausdorff content $\mathcal{H}^\delta_\infty$ is defined as

$$\int_{\mathbb{R}^n} |g(x)|^p \, d\mathcal{H}^\delta_\infty := p \int_0^\infty t^{p-1} \mathcal{H}^\delta_\infty(\{x \in \mathbb{R}^n : \; |g(x)| > t\}) \, dt.$$

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- Due to the monotonicity of set function $\mathcal{H}^\delta_\infty$, the Choquet integral is well defined for all functions, even not measurable with respect to Lebesgue measure.

- Unfortunately, this Choquet integral is not linear, even not sublinear:

$$\begin{aligned}\int_E f(x)\,d\mathcal{H}^\delta_\infty + \int_E g(x)\,d\mathcal{H}^\delta_\infty &\neq \int_E [f(x)+g(x)]\,d\mathcal{H}^\delta_\infty \\ &\leq 2\left[\int_E f(x)\,d\mathcal{H}^\delta_\infty + \int_E g(x)\,d\mathcal{H}^\delta_\infty\right].\end{aligned}$$

Equivalent dyadic version

- (Adams,1998) The dyadic Hausdorff content $\tilde{\mathcal{H}}_\infty^\delta$:

$$\tilde{\mathcal{H}}_\infty^\delta(K) := \inf\left\{\sum_{i=1}^\infty [l(Q_i)]^\delta : K \subset \bigcup_{i=1}^\infty Q_i\right\},$$

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- (Yang and Yuan, 2008) A new dyadic Hausdorff content:

$$\tilde{\mathcal{H}}_{\infty,0}^{\delta}(K) := \inf\left\{\sum_{i=1}^{\infty}[l(Q_i)]^{\delta} : K \subset \left(\bigcup_{i=1}^{\infty} Q_i\right)^{\circ}\right\},$$

  where $\{Q_i\}_i$ are dyadic cubes of $\mathbb{R}^n$.

- The Choquet integral with respect to $\tilde{\mathcal{H}}^{\delta}_{\infty,0}$ is **sub-linear**. By this, we have

$$\int_E \sum_{j\in\mathbb{N}} f_j(x)\, d\mathcal{H}^{\delta}_{\infty} \le C(n,\delta) \sum_{j\in\mathbb{N}} \int_E f_j(x)\, d\mathcal{H}^{\delta}_{\infty}.$$

The Hausdorff content and Choquet integral have many applicaitions in

- Morrey space, Besov-Triebel-Lizorkin type space and Q_α space$\cdots$
- Harmonic analysis and nonlinear potential theory; see Adams and L. Hedberg, Function Spaces and Potential Theory, Grundlehren, 314. Springer-Verlag, Berlin, 1996
- Quasilinear elliptic equations; see, for example, Kilpeläinen-Malý, Acta Math. 172 (1994), and Labutin, Duke Math. J. 111 (2002)
- Continuous time dynamic and coherent risk measures in finance; see, for example, Denis-Hu-Peng, Potential Anal. 34 (2011)
- Bayesian decision theory, subjective probability and robust optimization; see, for example, Bertsimas-Brown-Caramanis, SIAM Rev. 53 (2011).

2. Our purpose—what do we want to do?

In this talk, we are interested in identifying the conditions on the function w such that the weighted norm inequality

$$\int_{\mathbb{R}^n} [\mathcal{M}(f)(x)]^p \, w(x) \, dx \lesssim \int_{\mathbb{R}^n} |f(x)|^p \, w(x) \, dx,$$

remains true when Lebesgue measure dx is replaced by Hausdorff contents $d\mathcal{H}_\infty^\delta$, i.e., in the Choquet integral setting.

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remains true when Lebesgue measure dx is replaced by Hausdorff contents $d\mathcal{H}^\delta_\infty$, i.e., in the Choquet integral setting.

Then, we are further devoted to build up some important properties of the class of these functions w, such as

- the self-improving property, see Muckenhoupt, TAMS, 1972
- reverse Hölder inequality, see Coifman-Fefferman, Studia Math., 1974
- Jones' factorization theorem, see Jones, Ann. of Math., 1980.

Formulate the question

More precisely, the first purpose of this talk is to characterize the weight w on $\mathbb{R}^n$ such that there exists a positive constant K satisfying

$$\int_{\mathbb{R}^n} \left[\mathcal{M}_{\mathcal{H}^\delta_\infty}(f)(x)\right]^p w(x)\, d\mathcal{H}^\delta_\infty \le K \int_{\mathbb{R}^n} |f(x)|^p w(x)\, d\mathcal{H}^\delta_\infty, \tag{2.1}$$

when $p \in (1, \infty)$, and when $p = 1$, for any $t \in (0, \infty)$,

$$w_{\mathcal{H}^\delta_\infty}\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty}(f)(x) > t\right\} \le \frac{K}{t} \int_{\mathbb{R}^n} |f(x)| w(x)\, d\mathcal{H}^\delta_\infty. \tag{2.2}$$

Here and thereafter,

$$w_{\mathcal{H}^\delta_\infty}(F) := \int_F w(x)\, d\mathcal{H}^\delta_\infty, \quad \forall\, F \subset \mathbb{R}^n, \tag{2.3}$$

and $\mathcal{M}_{\mathcal{H}^\delta_\infty}$ is the capacitary Hardy-Littlewood maximal operator with respect to $\mathcal{H}^\delta_\infty$ defined as

$$\mathcal{M}_{\mathcal{H}^\delta_\infty} f(x) := \sup_{Q \ni x} \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q |f(x)|\, d\mathcal{H}^\delta_\infty.$$

First try

To settle this problem, it might seem intuitive and immediate to adapt the aforementioned methods for the classical A_p weight theory to the setting of Choquet integrals and capacitary maximal operators. However, upon closer examination, one can find that these approaches are not feasible and the thing is far away from immediate.

First try

A fundamental difficulty lies in the absence of linearity for the Choquet integrals, that is, for any non-negative functions $\{f_j\}_{j\in\mathbb{N}}$ defined on E,

$$\int_E \sum_{j\in\mathbb{N}} f_j(x)\, d\mathcal{H}^\delta_\infty \neq \sum_{j\in\mathbb{N}} \int_E f_j(x)\, d\mathcal{H}^\delta_\infty.$$

These two quantities are even not equivalent. Indeed, for any $M > 0$, there exist non-negative functions $\{f_j\}_{j\in\mathbb{N}}$ and E, such that

$$\sum_{j\in\mathbb{N}} \int_E f_j(x)\, d\mathcal{H}^\delta_\infty > M \int_E \sum_{j\in\mathbb{N}} f_j(x)\, d\mathcal{H}^\delta_\infty.$$

This makes that the aforementioned crucial techniques such as, self-improving property $A_p \Rightarrow A_{p-\varepsilon}$, reverse Hölder inequality and the Calderón-Zygmund decomposition, can not be extended to the current Hausdorff content and the Choquet integral setting directly. Consequently, new techniques and approaches are required.

3. Main Results

Capacitary Muckenhoupt's theorem

Theorem (Huang-Zhang-Z., 2025)

Let $\delta \in (0, n]$, $p \in (1, \infty)$ *and* w *be a weight. Then the following statements are equivalent*

(i) *the strong-type* (p, p) *inequality* (2.1) *holds;*

(ii) *there exists a positive constant* K *such that*

$$w_{\mathcal{H}^\delta_\infty}\left\{x \in \mathbb{R}^n : \mathcal{M}_{\mathcal{H}^\delta_\infty}(f)(x) > t\right\} \le \frac{K}{t^p} \int_{\mathbb{R}^n} |f(x)|^p w(x)\, d\mathcal{H}^\delta_\infty, \quad \forall\, t \in (0, \infty);$$

(iii) $w \in \mathcal{A}_{p,\delta}$, *i.e.*,

$$[w]_{\mathcal{A}_{p,\delta}} := \sup_{\text{cube } Q \subset \mathbb{R}^n} \left\{\frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q w(x)\, d\mathcal{H}^\delta_\infty\right\} \left\{\frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q w(x)^{-\frac{1}{p-1}}\, d\mathcal{H}^\delta_\infty\right\}^{p-1} < \infty. \tag{3.1}$$

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A weight w satisfying (3.1) with $p \in (1, \infty)$ is called the capacitary Muckenhoupt $\mathcal{A}_{p,\delta}$-weight with respect to the Hausdorff content $\mathcal{H}^\delta_\infty$, denoted as $w \in \mathcal{A}_{p,\delta}$.

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Theorem (Huang-Zhang-Z., 2025)

Let $\delta \in (0, n]$ *and* w *be a weight. Then the following statements are equivalent*

(i) *the weak-type* $(1, 1)$ *inequality* (2.2) *holds;*

(ii) $w \in \mathcal{A}_{1,\delta}$, *i.e.*

$$[w]_{\mathcal{A}_{1,\delta}} := \inf \left\{ K \in (0, \infty) : \ \mathcal{M}_{\mathcal{H}^\delta_\infty} w(x) \leq K w(x) \text{ for } \mathcal{H}^\delta_\infty\text{–a. e.} \right\} < \infty. \tag{3.2}$$

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A weight w satisfying (3.2) is called the capacitary Muckenhoupt $\mathcal{A}_{1,\delta}$-weight with respect to $\mathcal{H}^\delta_\infty$, denoted as $w \in \mathcal{A}_{1,\delta}$.

A corollary

As an application, we obtain the following weighted norm inequalities for classical Hardy-Littlewood maximal operators $\mathcal{M}$ on Choquet integrals by an interpolation argument. Let $\delta \in (0, n]$.

Corollary (Huang-Zhang-Z., 2025)

(i) *If* $w \in \mathcal{A}_{p,\delta}$ *with* $p \in (1, \infty)$ *and* $q \in [\frac{p\delta}{n}, \infty)$, *then*

$$\int_{\mathbb{R}^n} |\mathcal{M}f(x)|^q w(x)\, d\mathcal{H}^\delta_\infty \leq K \int_{\mathbb{R}^n} |f(x)|^q w(x)\, d\mathcal{H}^\delta_\infty.$$

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(iii) *If* $w \in \mathcal{A}_{p,\delta}$ *with* $p \in [1, \infty)$ *and* $q \in [\frac{p\delta}{n}, \infty)$, *then*

$$w_{\mathcal{H}^\delta_\infty}(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > t\}) \leq \frac{K}{t^q} \int_{\mathbb{R}^n} |f(x)|^q w(x)\, d\mathcal{H}^\delta_\infty.$$

An example of $\mathcal{A}_{p,\delta}$

Capacitary Muckenhoupt weight class $\mathcal{A}_{p,\delta}$

$$[w]_{\mathcal{A}_{p,\delta}} := \sup_{\text{cube } Q \subset \mathbb{R}^n} \left\{ \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q w(x)\, d\mathcal{H}^\delta_\infty \right\} \left\{ \frac{1}{\mathcal{H}^\delta_\infty(Q)} \int_Q w(x)^{-\frac{1}{p-1}}\, d\mathcal{H}^\delta_\infty \right\}^{p-1}$$

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When $\delta = n$, the weight class $\mathcal{A}_{p,\delta}$ goes back to classical Muckenhoupt weight class A_p.

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Proposition

Let $\delta \in (0, n]$ and $p \in [1, \infty)$. For any given $\alpha \in \mathbb{R}$, define $w(x) := |x|^{\alpha}$, $\forall\, x \in \mathbb{R}^n$. Then $w \in \mathcal{A}_{p,\delta}$ if and only if $\alpha \in (-\delta, \delta(p-1))$.

Monotonicity of capacitary Muckenhoupt weight class

- $\mathcal{A}_{p,\delta}$ is monotonically increasing with respect to the parameter p, i.e., $\mathcal{A}_{p_1,\delta} \subset \mathcal{A}_{p_2,\delta}$ when $1 \leq p_1 \leq p_2 < \infty$.

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Proposition

Let $0 < \delta < \beta \leq n$ *and* $p \in [1, \infty)$. *Then*

$$\mathcal{A}_{p,\delta} \subsetneqq \mathcal{A}_{p,\,\beta}$$

and $[w]_{\mathcal{A}_{p,\,\beta}} \leq K[w]_{\mathcal{A}_{p,\delta}}$.

- For any $\delta \in (0, n)$,

$$\mathcal{A}_{p,\delta} \subsetneqq \mathcal{A}_{p,\,n} = A_p$$

Remarks

We point out that our theorems go back to the classical Muckenhoupt's theroem.

- when $\delta = n$, the Hausdorff content $\mathcal{H}^n_\infty$ is equivalent to the Lebesgue measure
- capacitary maximal operator $\mathcal{M}_{\mathcal{H}^n_\infty}$ is just the classical maximal operator $\mathcal{M}$;
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To show the above two theorems, we develop a new approach in the Choquet integral setting where the classical methods no longer applicable.

Moreover, even when returning to the classical A_p weight setting, the approach developed provides new and broadly applicable proofs, which avoid linearity of integrals, the countable additivity of measures, or the Fubini theorem.

4. Main difficulty in proofs

The idea— part I

(i) In the setting of Lebesgue measure dx, via the Fubini theorem, for a weight $w(x)$ and $f \in L^1_w(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)| w(x)\, dx = \int_{\mathbb{R}^n} |f(x)|\, dw.$$

The idea— part I

(i) In the setting of Lebesgue measure dx, via the Fubini theorem, for a weight $w(x)$ and $f \in L^1_w(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f(x)| w(x)\, dx = \int_{\mathbb{R}^n} |f(x)|\, dw.$$

But, if $0 < \delta < n$, the following kind of the Fubini theorem

$$\int_{\mathbb{R}^n} \int_0^\infty f(x,t) w(x)\, dt\, d\mathcal{H}^\delta_\infty \sim \int_0^\infty \int_{\mathbb{R}^n} f(x,t) w(x)\, d\mathcal{H}^\delta_\infty\, dt.$$

fails in general, and hence the equivalence

$$\int_{\mathbb{R}^n} |f(x)| w(x)\, d\mathcal{H}^\delta_\infty \sim \int_{\mathbb{R}^n} |f(x)|\, dw_{\mathcal{H}^\delta_\infty} \tag{4.1}$$

does not hold for general weight w.

The idea— part I

Surprisingly, given $p \in [1, \infty)$, we discover, in the proposition below, that the condition $w \in \mathcal{A}_{p,\delta}$ ensures (4.1).

Proposition

Let $\delta \in (0, n]$, $p \in [1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$. Then there exists a positive constant $K(p)$ such that, for any $f \in L^1_w(\mathbb{R}^n, \mathcal{H}^\delta_\infty)$,

$$\frac{1}{4} \int_{\mathbb{R}^n} |f(x)| w(x)\, d\mathcal{H}^\delta_\infty \leq \int_{\mathbb{R}^n} |f(x)|\, dw_{\mathcal{H}^\delta_\infty} \leq K(p)[w]^{\frac{1}{p}}_{\mathcal{A}_{p,\delta}} \int_{\mathbb{R}^n} |f(x)| w(x)\, d\mathcal{H}^\delta_\infty.$$

This proposition plays a pivotal role throughout the proof, which partly fills the gap left by the absence of the Fubini theorem in the weighted Choquet integral setting.

The idea — part II

(ii) Inspired by the idea from Calderón-Zygmund decomposition technique, we prove a "sparse covering lemma" in the context of Hausdorff contents as follows.

Proposition

Let $\delta \in (0, n]$ and E be a subset of $\mathbb{R}^n$ satisfying $\widetilde{\mathcal{H}}^\delta_\infty(E) < \infty$. Then there exists a subset $F \subset \mathbb{R}^n$ and a family $\{Q_j\}_{j\in\mathbb{N}}$ of non-overlapping dyadic cubes in $\mathbb{R}^n$ such that

(i) $E \subset \left(\bigcup_{j\in\mathbb{N}} Q_j\right) \cup F$ *and* $\widetilde{\mathcal{H}}^\delta_\infty(F) = 0$;

(ii) $\sum_{j\in\mathbb{N}} \widetilde{\mathcal{H}}^\delta_\infty(Q_j) \le 2\widetilde{\mathcal{H}}^\delta_\infty(E)$;

(iii) *for any* $j \in \mathbb{N}$, *we have* $\widetilde{\mathcal{H}}^\delta_\infty(Q_j) \le 3\widetilde{\mathcal{H}}^\delta_\infty(Q_j \cap E)$.

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- This proposition seems to be new even when reduced to the setting of classical Lebesgue measure, i.e., in the case of $\delta = n$.
- It serves as a partial substitute for the linearity property, which generally fails to hold for the Choquet integral.

Applying the "sparse covering lemma", we obtain the following conclusion, which realizes the interchange of summation and integration with respect to $\mathcal{H}_\infty^\delta$ in a certain sense.

Lemma

Let $\delta \in (0, n]$, $p \in [1, \infty)$, $E \subset \mathbb{R}^n$ *and* $w \in \mathcal{A}_{p,\delta}$ *satisfying*

$$\int_E w(x)\, d\mathcal{H}_\infty^\delta < \infty.$$

Then there exist a family $\{Q_j\}_{j\in\mathbb{N}}$ *of non-overlapping dyadic cubes and a subset* $F \subset \mathbb{R}^n$ *with* $\mathcal{H}_\infty^\delta(F) = 0$ *such that* $E \subset (\bigcup_{j\in\mathbb{N}} Q_j) \bigcup F$ *and*

$$\sum_{j\in\mathbb{N}} \int_{Q_j} w(x)\, d\mathcal{H}_\infty^\delta \leq K(n,\delta,p)[w]_{\mathcal{A}_{p,\delta}} \int_E w(x)\, d\mathcal{H}_\infty^\delta.$$

The idea — part II

Moreover, the following conclusion, with a weighted packing condition for a family $\{Q_j\}_j$ of non-overlapping cubes, becomes essential. It serves as a partial substitute for the linearity property.

Proposition

Let $\delta \in (0, n]$, $p \in [1, \infty)$ and $w \in \mathcal{A}_{p,\delta}$. Let $\{Q_j\}_{j\in\mathbb{N}}$ be a family of non-overlapping dyadic cubes of $\mathbb{R}^n$. If there exists a constant $\beta > 0$ such that, for each dyadic cube Q,

$$\sum_{Q_j \subset Q} w_{\mathcal{H}^\delta_\infty}(Q_j) \le \beta\, w_{\mathcal{H}^\delta_\infty}(Q),$$

then

$$\sum_{j\in\mathbb{N}} \int_{Q_j} |f(x)| w(x)\, d\mathcal{H}^\delta_\infty \le \max\{1, \beta\} [w]^{1+\frac{1}{p}}_{\mathcal{A}_{p,\delta}} \int_{\bigcup_{j\in\mathbb{N}} Q_j} |f(x)| w(x)\, d\mathcal{H}^\delta_\infty.$$

5. Applications to theory of capacitary Muckenhoupt weight class.

As a result of the weighted norm inequality, we obtain the Jones factorization for $\mathcal{A}_{p,\delta}$

Theorem

Let $\delta \in (0, n]$ *and* $p \in [1, \infty)$*. Then* $w \in \mathcal{A}_{p,\delta}$ *if and only if there exist two weights* $w_0,\ w_1 \in \mathcal{A}_{1,\delta}$ *such that* $w = w_0 w_1^{1-p}$*.*

The reverse Hölder inequality for $\mathcal{A}_{p,\delta}$

Theorem

Let $\delta \in (0, n]$, $p \in [1, \infty)$ *and* $w \in \mathcal{A}_{p,\delta}$. *Then there exists positive constants* $K = K(n, \delta, p, [w]_{\mathcal{A}_{p,\delta}})$ *and* $\gamma = \gamma(n, \delta, p, [w]_{\mathcal{A}_{p,\delta}}) \in (0, 1)$ *such that, for every cube* Q,

$$\left[\frac{1}{\mathcal{H}^{\delta}_{\infty}(Q)} \int_Q w(x)^{1+\gamma} \, d\mathcal{H}^{\delta}_{\infty}\right]^{\frac{1}{1+\gamma}} \leq \frac{K}{\mathcal{H}^{\delta}_{\infty}(Q)} \int_Q w(x) \, d\mathcal{H}^{\delta}_{\infty}. \tag{5.1}$$

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Corollary

Let $\delta \in (0, n]$, $p \in [1, \infty)$ *and* $w \in \mathcal{A}_{p,\delta}$. *Then there exists a constant* $\gamma \in (0, 1)$ *such that* $w^{1+\gamma} \in \mathcal{A}_{p,\delta}$.

The self-improving property for $\mathcal{A}_{p,\delta}$

Theorem

Let $\delta \in (0, n]$, $p \in (1, \infty)$ *and* $w \in \mathcal{A}_{p,\delta}$. *Then there is a* q *with* $1 < q < p$ *such that* $w \in \mathcal{A}_{q,\delta}$.

Remarks

- The above theorems on capacitary Muckenhoupt weight class are proved with the help of capacitary Muckenhoupt's theorem and the new methods. This is different with the classical cases, in which these properties imply Muckenhoupt's theorem.

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- When $\delta = n$, the above theorems return exactly to classical theory of Muckenhoupt weight class A_p, that is,
  - the self-improving property, see Muckenhoupt, TAMS, 1972
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  - Jones' factorization theorem, see Jones, Ann. of Math., 1980.
- When $\delta \in (0, n)$, the classical methods no longer apply. In this case, our proofs avoid linearity of integrals, the countable additivity of measures, or the Fubini theorem.

Conclusion

- Introduce a novel capacitary Muckenhoupt weight class and build up the Muckenhoupt's theorem.

- Develop a new approach in the Choquet integral setting where the classical methods no longer apply, avoiding linearity of integrals, the countable additivity of measures, or the Fubini theorem.

- Extend the classical theory beyond measure-theoretic frameworks.

- Establish some properties of capacitary Muckenhoupt weight class

    the strict monotonicity on the dimension index δ;

    the Jones factorization theorem;

    the reverse Hölder inequality;

    the self-improving property.

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Thank you!