Lipschitz空间函数到其子空间距离的实变特征刻画及其应用

北京师范大学数学科学学院 2025年12月

动机

- ▶ 1978, Garnett and Jones [gj78] 建立$\mathbb{R}^n$中 BMO 函数到子空间$L^\infty(\mathbb{R}^n)$距离的差分刻画, 作为应用给出了 $L^\infty(\mathbb{R}^n)$ 在 BMO $(\mathbb{R}^n)$中闭包的差分刻画.

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光滑函数在BMO空间的闭包问题及应用

- D. Sarason, Functions of vanishing mean oscillation, Trans. Amer. Math. Soc. 207 (1975), 391–405.
- F. Chiarenza, M. Frasca and P. Longo, $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc. 336 (1993), 841–853.
- Z. Shen, Bounds of Riesz transforms on L^p spaces for second order elliptic operators, Ann. Inst. Fourier (Grenoble) 55 (2005), 173–197.

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▶Lipschitz空间: all measurable functions f on $\mathbb{R}^n$ for which there exists a continuous function g on $\mathbb{R}^n$ such that $f(x) = g(x)$ almost everywhere on $\mathbb{R}^n$ and

$$\|f\|_{\Lambda_s(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{|D_h^{\lfloor s\rfloor+1}(g)(x)|}{|h|^s} < \infty,$$

where $s \in (0,\infty)$, $\lfloor s\rfloor$ denotes the *largest integer not greater than* s, $\mathbf{0}$ is the *origin* of $\mathbb{R}^n$, and, for any $r \in \mathbb{N}$, D_h^r denotes the *r-th order difference operator* with step $h \in \mathbb{R}^n$ given by

$$D_h(f) := f(\cdot + h) - f \text{ and } D_h^{r+1} := D_h^r(D_h).$$

定义中差分项真正起作用的是步长比较小的时候, 步长比较大的时候函数的有界性在起作用, 光滑性是由步长趋于零时函数差分的衰减速度所体现的.

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▶问题: 建立Lipschitz空间函数到其子空间距离的实变特征并给出这些子空间在Lipschitz空间中闭包的实变特征.

▶ 例如: Sobolev 空间, Besov 空间, Triebel–Lizorkin空间?

▶为什么选择Lipschitz空间:涵盖所有正的光滑性指标, 大多数空间在Lipschitz空间中不稠密.

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▶关于Lipschitz空间的结果: Nicolau 和 Soler i Gibert [ns]用差分 刻画 $\mathrm{dist}\,(f, I_1(\mathrm{BMO})(\mathbb{R}))_{\Lambda_1(\mathbb{R})}$对于所有有紧支集的$f \in \Lambda_1(\mathbb{R})$.

这篇文章中的方法只适用于1维

▶ $I_1(\mathrm{BMO})(\mathbb{R})$, consisting of all continuous functions with distributional derivatives in $\mathrm{BMO}(\mathbb{R})$, in terms of the second-order differences. Recall that $I_1(\mathrm{BMO})(\mathbb{R}^n)$ is also known to be the image of $\mathrm{BMO}(\mathbb{R}^n)$ under the first-order Riesz potential I_1.

[ns]A. Nicolau and O. Soler i Gibert , Approximation in the Zygmund class, J. Lond. Math. Soc. (2) 101 (2020), 226–246.

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- Saksman 和 Soler i Gibert [ss]用差分刻画 $\operatorname{dist}(f, F^s_{\infty,2}(\mathbb{R}^n))_{\Lambda_s(\mathbb{R}^n)}$, $s \in (0,1]$, $f \in \Lambda_s(\mathbb{R}^n)$.

将维数推广到了n维, 证明方法是基于小波的.

[ss]E. Saksman and O. Soler i Gibert, Approximation in the Zygmund and Hölder classes on $\mathbb{R}^n$, Canad. J. Math. 74 (2022), 1745–1770.

- 研究的问题：对于所有光滑指标$s \in (0,\infty)$, 给出Lipschitz空间中的元素到具有小波刻画的函数空间的距离的实变特征刻画并给出相应的闭包的实变刻画.

主要结果:小波

- $C^r_{\mathrm{c}}(\mathbb{R}^n)$, the set of all r times continuously differentiable real-valued functions on $\mathbb{R}^n$ with compact support.
- For any $\gamma := (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n_+$ and $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n}$.
- For any integer $L > 1$, there exist functions $\varphi, \psi_\ell \in C^L_{\mathrm{c}}(\mathbb{R}^n)$ with $\ell \in \{1, \ldots, 2^n - 1\} =: \mathcal{N}$ such that, for any $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| \le L$,

$$\int_{\mathbb{R}} \psi_\ell(x) x^\gamma \, dx = 0, \quad \forall\, \ell \in \mathcal{N}$$

and the union of

$$\Phi := \{\varphi_k(x) := \varphi(x-k) : \; k \in \mathbb{Z}^n\} \tag{1}$$

and

$$\Psi := \left\{\psi_{\ell,j,k}(x) := 2^{jn/2}\psi_\ell(2^j x - k) : \; \ell \in \mathcal{N}, \; j \in \mathbb{Z}_+, \; k \in \mathbb{Z}^n\right\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^n)$. The system $\Phi \cup \Psi$ is called the *Daubechies wavelet system of regularity* L on $\mathbb{R}^n$.

主要结果:小波

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- For any integer $L > 1$, there exist functions $\varphi, \psi_\ell \in C_c^L(\mathbb{R}^n)$ with $\ell \in \{1, \ldots, 2^n - 1\} =: \mathcal{N}$ such that, for any $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq L$,

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forms an orthonormal basis of $L^2(\mathbb{R}^n)$. The system $\Phi \cup \Psi$ is called the *Daubechies wavelet system of regularity L* on $\mathbb{R}^n$.

主要结果:小波与二进方体

▶Let $\mathcal{D}$ be the class of all dyadic cubes $I_{j,k} := \{x \in \mathbb{R}^n : 2^j x - k \in [0,1)^n\}$ with $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^n$

▶ $\mathcal{D}_j$ denote the class of all dyadic cubes in $\mathcal{D}$ with edge lengths 2^{-j} with $j \in \mathbb{Z}_+$.

▶ Let $\Omega := \Omega_0 \cup \Omega_1$, where $\Omega_0 := \{0\} \times \mathcal{D}_0$ and $\Omega_1 := \mathcal{N} \times \mathcal{D}$.

▶ For any $\omega = (0, I) \in \Omega_0$ with $I = k + [0,1]^n \in \mathcal{D}_0$, let

$$\psi_\omega(x) := \varphi(x - k), \tag{2}$$

while, for any $\omega = (\ell_\omega, I_\omega) \in \Omega_1$ with $\ell_\omega \in \mathcal{N}$ and $I_\omega = 2^{-j}(k + [0,1]^n) \in \mathcal{D}_j$, let

$$\psi_\omega(x) := 2^{jn/2} \psi_{\ell_\omega}(2^j x - k). \tag{3}$$

▶For any $f \in \Lambda_s(\mathbb{R}^n)$ and $\omega \in \Omega$, let $\langle f, \psi_\omega \rangle := \int_{\mathbb{R}^n} f(x)\psi_\omega(x)\,dx$.

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主要结果:小波特征空间

▶Let X be a quasi-normed lattice of function sequences. Assume that $s \in (0, \infty)$. The *Daubechies s-Lipschitz X-based space* $\Lambda_X^s(\mathbb{R}^n)$ is defined to be the set of all functions $f \in \Lambda_s(\mathbb{R}^n)$ such that

$$\|f\|_{\Lambda_X^s(\mathbb{R}^n)} := \left\| \left\{ \sum_{\omega\in\Omega, I_\omega\in\mathcal{D}_j} |I_\omega|^{-s/n-1/2} |\langle f, \psi_\omega\rangle| \mathbf{1}_{I_\omega} \right\}_{j\in\mathbb{Z}_+} \right\|_X < \infty.$$

▶ : $M(\mathbb{R}^n)$, the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ that are finite almost everywhere on $\mathbb{R}^n$.

▶$M_{\mathbb{Z}_+}(\mathbb{R}^n)$ the set of all sequences $\{f_j\}_{j\in\mathbb{Z}_+}$ of functions in $M(\mathbb{R}^n)$.

▶$\|\cdot\|_X : M_{\mathbb{Z}_+}(\mathbb{R}^n) \to [0, \infty]$

▶ X中的每个元素都是一列函数, $\mathcal{D}_j$是第j层二进方体, ψ_ω是非齐次的小波系统.

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主要结果:小波特征空间

▶Let $q, p \in (0, \infty]$. The *space* $l^q(L^p)_{\mathbb{Z}_+}(\mathbb{R}^n)$ is defined to be the set of all $\mathbf{F} := \{f_j\}_{j\in\mathbb{Z}_+} \in M_{\mathbb{Z}_+}(\mathbb{R}^n)$ such that

$$\|\mathbf{F}\|_{l^q(L^p)_{\mathbb{Z}_+}(\mathbb{R}^n)} := \left[\sum_{j\in\mathbb{Z}_+} \|f_j\|^q_{L^p(\mathbb{R}^n)}\right]^{\frac{1}{q}} < \infty,$$

where the usual modification is made when $q = \infty$.

▶ : $B^s_{p,q}(\mathbb{R}^n) \cap \Lambda_s(\mathbb{R}^n) = \Lambda^s_X(\mathbb{R}^n)$ with equivalent quasi-norm, $X := l^q(L^p)_{\mathbb{Z}_+}(\mathbb{R}^n)$.

主要结果:小波特征空间

- Let $q, p \in (0, \infty]$. $L^p(l^q)_{\mathbb{Z}_+}(\mathbb{R}^n)$, the set of all $\mathbf{F} := \{f_j\}_{j\in\mathbb{Z}_+} \in M_{\mathbb{Z}_+}(\mathbb{R}^n)$ such that

$$\|\mathbf{F}\|_{L^p(l^q)_{\mathbb{Z}_+}(\mathbb{R}^n)} := \left\| \left[\sum_{j\in\mathbb{Z}_+} |f_j|^q \right]^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where the usual modification is made when $q = \infty$.

- $F_{\infty,q}(\mathbb{R}^n, \mathbb{Z}_+)$, the set of all $\mathbf{F} := \{f_j\}_{j\in\mathbb{Z}_+} \in M_{\mathbb{Z}_+}(\mathbb{R}^n)$ such that

$$\|f\|_{F_{\infty,q}(\mathbb{R}^n,\mathbb{Z}_+)} := \sup_{l\in\mathbb{Z}_+, m\in\mathbb{Z}^n} \left\{ \frac{1}{|I_{l,m}|} \int_{I_{l,m}} \sum_{j=l}^{\infty} |f_j(x)|^q \, dx \right\}^{\frac{1}{q}} < \infty,$$

where the usual modification is made when $q = \infty$.

主要结果:小波特征空间

- $F^s_{p,q}(\mathbb{R}^n) \cap \Lambda_s(\mathbb{R}^n) = \Lambda^s_X(\mathbb{R}^n)$, $X := L^p(l^q)_{\mathbb{Z}_+}(\mathbb{R}^n)$ when $p \neq \infty$ or with $X := F_{\infty,q}(\mathbb{R}^n, \mathbb{Z}_+)$ when $p = \infty$.

主要结果

- Let $s \in (0, \infty)$. Assume that the regularity parameter $L \in \mathbb{N}$ of the Daubechies wavelet system $\{\psi_\omega\}_{\omega \in \Omega}$ satisfies that $L > s$. Then, for any $f \in \Lambda_s(\mathbb{R}^n)$,

$$\mathrm{dist}\,(f, \Lambda_X^s(\mathbb{R}^n))_{\Lambda_s(\mathbb{R}^n)} \sim \inf \left\{ \varepsilon \in (0, \infty) : \left\| \left\{ \sum_{I \in W_j^0(s,f,\varepsilon)} \mathbf{1}_I \right\}_{j \in \mathbb{Z}_+} \right\|_X < \infty \right\}$$

with positive equivalence constants independent of f.

$$W^0(s, f, \varepsilon) := \left\{ I \in \mathcal{D} : \max_{\omega \in \Omega, I_\omega = I} |\langle f, \psi_{(\ell,I)} \rangle| |I|^{-\frac{s}{n} - \frac{1}{2}} > \varepsilon \right\}$$

主要结果

- 有了小波特征的特征刻画, 就建立了这个距离与实变特征的桥梁.
- 利用小波特征, 我们便可以利用小波与各个实变特征的联系, 进一步建立其他实变特征.
- 我们后续得到了差分特征, 半群特征, 球平均特征, 乘子特征.

证明思想

- Lipschitz空间函数的小波特征是小波系数乘光滑指标的上确界.

$$\|f\|_{\Lambda_s(\mathbb{R}^n)} \sim \sup_{\omega\in\Omega} |\langle f,\psi_\omega\rangle|\,|I_\omega|^{-s/n-1/2}.$$

- 利用小波特征, Lipschitz空间函数f与函数g的距离就是对应系数之差的上确界.

$$\sup_{\omega\in\Omega} |\langle f-g,\psi_\omega\rangle||I_\omega|^{-s/n-1/2}$$

- Lipschitz空间函数到其子空间$\Lambda_X^s(\mathbb{R}^n)$的距离是.

$$\inf_{g\in\Lambda_X^s(\mathbb{R}^n)} \|f-g\|_{\Lambda_s(\mathbb{R}^n)} \sim \inf_{g\in\Lambda_X^s(\mathbb{R}^n)} \sup_{\omega\in\Omega} |\langle f-g,\psi_\omega\rangle||I_\omega|^{-s/n-1/2}.$$

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- Lipschitz空间函数的小波特征是小波系数乘光滑指标的上确界.

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证明思想

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▶Lipschitz空间函数到其子空间$\Lambda_X^s(\mathbb{R}^n)$的距离是.

$$\inf_{g\in\Lambda_X^s(\mathbb{R}^n)}\|f-g\|_{\Lambda_s(\mathbb{R}^n)}\sim\inf_{g\in\Lambda_X^s(\mathbb{R}^n)}\sup_{\omega\in\Omega}|\langle f-g,\psi_\omega\rangle||I_\omega|^{-s/n-1/2}.$$

▶接下来的任务是去掉下确界,也就是尽最大可能去找到空间$\Lambda_X^s(\mathbb{R}^n)$中可以在Lip范数下逼近$f$的元素.

▶观察f的小波特征, 将f去掉什么才能缩小它的范数呢?

$$\|f\|_{\Lambda_s(\mathbb{R}^n)}\sim\sup_{\omega\in\Omega}|\langle f,\psi_\omega\rangle|\,|I_\omega|^{-s/n-1/2}.$$

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- 因为一个函数的lip范数是完全由最大系数决定的, 如果想要降低一个函数lip范数, 比如想让这个lip范数小于1, 那么就要把大于1的所有系数拿走. 只要剩下一个系数>1, 这个lip范数就会 > 1.

- 通过保留所有较大的系数实现的逼近就是贪婪算法.

证明思想

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▶通过保留所有较大的系数实现的逼近就是贪婪算法.

For any $\varepsilon \in (0, \infty)$, let

$$G_\varepsilon(f) := \sum_{\omega\in\Omega(\varepsilon)} \langle f, \psi_\omega\rangle \psi_\omega,$$

其中$\Omega(\varepsilon)$就是所有系数大于ε的指标集,如下:

$$W^0(s, f, \varepsilon) := \left\{ I \in \mathcal{D} : \max_{\omega\in\Omega, I_\omega = I} |\langle f, \psi_{(\ell,I)}\rangle| |I|^{-\frac{s}{n}-\frac{1}{2}} > \varepsilon \right\}$$

于是立刻有

$$\|f - G_\varepsilon(f)\|_{\Lambda_s(\mathbb{R}^n)} \lesssim \sup_{\omega\in\Omega\setminus\Omega(\varepsilon)} |\langle f, \psi_\omega\rangle| |I_\omega|^{-s/n-1/2} \leq \varepsilon.$$

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此时我们已经找到了一个逼近方法, 通过这个方法我们可以让逼近误差ε任意的小.

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►接下来的问题是$G_\varepsilon(f)$何时在我们的目标空间$\Lambda^s_X(\mathbb{R}^n)$中？
►$G_\varepsilon(f)$选取的都是系数大于ε的部分，而这些系数本身又是小于等于$\|f\|_{\Lambda_s(\mathbb{R}^n)}$,因为$f$的小波特征就是系数的上确界. 即:

$$\varepsilon < |I_\omega|^{-\frac{s}{n}-\frac{1}{2}} |\langle G_\varepsilon(f), \psi_\omega\rangle| \lesssim \|f\|_{\Lambda_s(\mathbb{R}^n)}.$$

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由$\varepsilon < |I_\omega|^{-\frac{s}{n}-\frac{1}{2}} |\langle G_\varepsilon(f), \psi_\omega\rangle| \lesssim \|f\|_{\Lambda_s(\mathbb{R}^n)}$推出

$$\varepsilon \left\| \left\{ \sum_{I\in W_j^0(s,f,\varepsilon)} \mathbf{1}_I \right\}_{j\in\mathbb{Z}_+} \right\|_X \lesssim \|G_\varepsilon(f)\|_{\Lambda_X^s(\mathbb{R}^n)}$$

$$= \left\| \left\{ \sum_{\omega\in\Omega, I_\omega\in W_j^0(s,f,\varepsilon)} |I_\omega|^{-\frac{s}{n}-\frac{1}{2}} |\langle G_\varepsilon(f), \psi_\omega\rangle| \mathbf{1}_{I_\omega} \right\}_{j\in\mathbb{Z}_+} \right\|_X$$

$$\lesssim \|f\|_{\Lambda_s(\mathbb{R}^n)} \left\| \left\{ \sum_{I\in W_j^0(s,f,\varepsilon)} \mathbf{1}_I \right\}_{j\in\mathbb{Z}_+} \right\|_X .$$

证明思想

▶接下来的问题是$G_\varepsilon(f)$何时在我们的目标空间$\Lambda_X^s(\mathbb{R}^n)$中？

$$\varepsilon \left\| \left\{ \sum_{l\in W_j^0(s,f,\varepsilon)} \mathbf{1}_l \right\}_{j\in\mathbb{Z}_+} \right\|_X \lesssim \|G_\varepsilon(f)\|_{\Lambda_X^s(\mathbb{R}^n)}$$

$$\lesssim \|f\|_{\Lambda_s(\mathbb{R}^n)} \left\| \left\{ \sum_{l\in W_j^0(s,f,\varepsilon)} \mathbf{1}_l \right\}_{j\in\mathbb{Z}_+} \right\|_X .$$

▶发现$G_\varepsilon(f) \in \Lambda_X^s(\mathbb{R}^n)$当且仅当 $\|\{\sum_{l\in W_j^0(s,f,\varepsilon)} \mathbf{1}_l\}_{j\in\mathbb{Z}_+}\|_X < \infty$.

证明思想

▶总结一下目前得到的结论 对于Lip空间中任意的一个函数f.

$$\|f - G_\varepsilon(f)\|_{\Lambda_s(\mathbb{R}^n)} \leq \varepsilon.$$

并且

$G_\varepsilon(f) \in \Lambda_X^s(\mathbb{R}^n)$当且仅当 $\|\{\sum_{I \in W_j^0(s,f,\varepsilon)} \mathbf{1}_I\}_{j \in \mathbb{Z}_+}\|_X < \infty$.

证明思想

- $\|f - G_\varepsilon(f)\|_{\Lambda_s(\mathbb{R}^n)} \leq \varepsilon$.
  $G_\varepsilon(f) \in \Lambda_X^s(\mathbb{R}^n)$当且仅当 $\|\{\sum_{l \in W_j^0(s,f,\varepsilon)} \mathbf{1}_l\}_{j \in \mathbb{Z}_+}\|_X < \infty$.
- 等价性

$$
\begin{aligned}
&\operatorname{dist}\,(f, \Lambda_X^s(\mathbb{R}^n))_{\Lambda_s(\mathbb{R}^n)} \\
&\sim \inf \left\{ \varepsilon \in (0, \infty) : \left\| \left\{ \sum_{l \in W_j^0(s,f,\varepsilon)} \mathbf{1}_l \right\}_{j \in \mathbb{Z}_+} \right\|_X < \infty \right\}
\end{aligned}
$$

应用

Let $s \in (0,\infty)$, $p \in (0,\infty)$, and $q \in (0,\infty]$.

$$\begin{aligned}&\mathrm{dist}\,\left(f, B^s_{p,q}(\mathbb{R}^n)\cap \Lambda_s(\mathbb{R}^n)\right)_{\Lambda_s(\mathbb{R}^n)}\\ &\sim \inf\left\{\varepsilon\in(0,\infty):\ \left\{\sum_{j=0}^{\infty}\left[\sum_{I\in W^0(s,f,\varepsilon),I\in\mathcal{D}_j}|I|\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}<\infty\right\}\end{aligned}$$

应用

Let $s \in (0,\infty)$, $p \in (0,\infty)$, and $q \in (0,\infty]$.

$$
\begin{aligned}
&\operatorname{dist}\left(f, B^s_{p,q}(\mathbb{R}^n) \cap \Lambda_s(\mathbb{R}^n)\right)_{\Lambda_s(\mathbb{R}^n)} \\
&\sim \inf\left\{\varepsilon \in (0,\infty) : \left\{\sum_{j=0}^{\infty}\left[\mu(S_{r,j}(s,f,\varepsilon))\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}} < \infty\right\} \\
&+ \limsup_{k\in\mathbb{Z}^n,\ |k|\to\infty}\left|\int_{\mathbb{R}^n}\varphi(x-k)f(x)\,dx\right|
\end{aligned}
$$

► $j \in \mathbb{Z}_+$,

$$
S_{r,j}(s,f,\varepsilon) := \left\{(x,y) \in \mathbb{R}^{n+1}_+ :\ \Delta_r f(x,y) > \varepsilon y^s,\ y \in (2^{-j-1}, 2^{-j}]\right\}.
$$

Let $s \in (0, \infty)$ and $q \in (0, \infty]$.

$$\operatorname{dist}\left(f, B^s_{\infty,q}(\mathbb{R}^n)\right)_{\Lambda_s(\mathbb{R}^n)}$$
$$\sim \inf\left\{\varepsilon \in (0, \infty) : \ \sharp\{j \in \mathbb{Z}_+ : \ W_j(s, f, \varepsilon) \neq \emptyset\} < \infty\right\}$$

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谢 谢！