Normal numbers and measures

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Normal number

Definition (Borel, 1909)

Let $b \in \mathbb{N} \setminus \{1\}$*. A real number* $x = 0.a_1 a_2 \cdots = \sum_{k=1}^{\infty} a_k b^{-k} \in [0,1]$ *is normal to base* b *(or* b*-normal) if, and only if,* $\forall k \in \mathbb{N}$ *and any combination* $D_k = (d_1, \ldots, d_k)$ *of* k *digits,* $d_i \in \{0, 1, \ldots, b-1\}$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sharp \left\{0 \leq j \leq N-1 : (a_{j+1}, \ldots, a_{j+k}) = (d_1, \ldots, d_k)\right\} = \frac{1}{b^k}.$$

.

$$x = 0.2113461137732113\cdots,\ D_3 = (1,1,3),\ b = 10.$$

- É. Borel, Les probabilités dénombrables et leurs applications arithmétiques, Rend. Circ. Math. Palermo 27 (1909), 247-271.

Normal number

1) Rational numbers can not be normal to any base:

$$\frac{1}{7} = 0.142857142857\cdots \quad \text{(10-ary expansion)},$$

where only the combination of $D_6 = (1, 4, 2, 8, 5, 7)$ occurs infinitely often.

2) It is not known whether or not π, e, $\sqrt{2}$, ... are normal to a fixed base.

- Y. Bugeaud, Distribution modulo one and Diophantine approximation, Cambridge Tracts in Mathematics, 193. Cambridge University Press, Cambridge, 2012.

Normal number

Theorem (Borel, 1909)

Lebesgue almost every $x \in [0, 1]$ *is normal to every* $b \geq 2$*.*

Theorem (Strong law of large numbers)

Assume that $X_1, X_2, \ldots$ *are i.i.d. random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, which satisfy* $E(X_i) = \mu$*. Then it holds*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} X_n(\omega) = \mu, \quad a.e.\ \omega \in \Omega.$$

Proof of Borel's theorem

Let $\Omega := \{0,1\}^{\mathbb{N}} = [0,1]$ and $\mathbb{P}$ be the uniform infinite product probability measure on Ω (Lebesgue measure on $[0,1]$). For any $\omega = 0.\omega_1\omega_2\cdots \in \Omega$, let

$$X_n : \ \Omega \to \{0,1\}, \quad X_n(\omega) = \omega_n.$$

Then X_n are i.i.d. (Bernoulli), $\mathbb{P}(X_n = 0) = 1/2, \quad \mathbb{P}(X_n = 1) = 1/2$, and $E(X_n) = 1/2$.

Normal number

Normal numbers are everywhere, but it is difficult to find an explicit one.

- Champernowne [J. London Math. Soc., 1933]:

$$\xi_c := 0.1234567891011121314\cdots$$

is normal to base 10.

- Copeland and Erdös [Bull. Amer. Math. Soc., 1946]:

$$0.23571113171923293 1\cdots$$

is normal to base 10.

Normality & uniform distribution

Definition (Bohl, Sierpiński, Weyl)

A sequence $(x_n)_{n\geq 1}$ of real numbers is uniformly distributed modulo one if, for any $0 \leq \alpha < \beta \leq 1$, the fractional part of x_n satisfies

$$\lim_{N\to\infty} \frac{1}{N}\sharp\{1 \leq n \leq N\colon\ \{x_n\} \in [\alpha, \beta)\} = \beta - \alpha.$$

- P. Bohl, J. Reine Angew. Math., 135 (1909), 189-283.
- M.W. Sierpiński, Krakau Anz. A (1910), 9-11.
- H. Weyl, Rend. Circ. Mat. Palermo 30 (1910), 377-407.

Normality & uniform distribution

Theorem (Wall, 1949)

Let $b \in \mathbb{N}$ with $b \geq 2$. Then $x \in [0, 1]$ is normal to base b if, and only if, the sequence $(b^{n-1}x)_{n\geq 1}$ is uniformly distributed modulo one.

- D. D. Wall, Normal numbers. PhD thesis, University of California, Berkeley, CA, 1949.

Normality & uniform distribution

Theorem (Weyl, 1916)

The sequence $(x_n)_{n\geq 1}$ of real numbers is uniformly distributed modulo one, if and only if, for any $h \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{-2\pi i h x_n} = 0 \quad \textit{(Weyl's criterion)}.$$

- H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins., Math. Ann. 77 (1916), 313-352.

DEL theorem

Theorem (Davenport-Erdős-LeVeque, 1963)

Let μ be a finite Borel measure on $[0,1]$, and $\{s_n(x)\}_{n\geq 1}$ be a sequence of bounded and integrable functions defined on $[0,1]$. If, for any $h \in \mathbb{Z}\setminus\{0\}$,

$$\sum_{N=1}^{\infty} \frac{1}{N} \int_0^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i h s_n(x)} \right|^2 \mathrm{d}\mu(x) < \infty,$$

then, $\{s_n(x)\}_{n\geq 1}$ is uniformly distributed modulo one for μ-a.e. $x \in [0,1]$.

- H. Davenport, P. Erdős, & W. J. LeVeque, On Weyl's criterion for uniform distribution, Michigan Math. J. 10 (1963), 311-314.

DEL theorem

- For any integer $b \geq 2$ and $x \in [0,1]$, take $s_n(x) = b^n x$. If, for any $h \in \mathbb{Z} \setminus \{0\}$,

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |\widehat{\mu}\left(h(b^n - b^m)\right)| < \infty,$$

then μ-a.e. $x \in [0,1]$ is b normal, where

$$\widehat{\mu}(\xi) := \int_0^1 e^{-2\pi i \xi x} \mathrm{d}\mu(x), \quad \xi \in \mathbb{R}.$$

- It is better to work with measures than with sets.

Partly normal numbers

- Do there exist real numbers which are normal to one base r, but not normal to another base s ?
- [Steinhaus, New Scottish Book]: whether normality to infinitely many bases implies normality to all bases?
- Cantor set:

$$C_{1/3} = \left\{ x \in [0,1] : \; x = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j}, \quad \varepsilon_j \in \{0,2\} \right\}.$$

Every $x \in C_{1/3}$ is not normal to base 3.

Partly normal numbers

- Integers $r, s \geq 2$ are multiplicatively independent, $r \not\sim s$, if $\log r / \log s \notin \mathbb{Q}$, i.e. there exists no $m, n \in \mathbb{N}$ s.t. $r^m = s^n$.

Theorem (Cassels, 1959; Schmidt, 1960)

Let μ be the Cantor measure on $C_{1/3}$. Then μ-a.e. $x \in C_{1/3}$ is normal to base $b \not\sim 3$.

- J. W. S. Cassels, On a problem of Steinhaus about normal numbers, Colloq. Math. 7 (1959), 95-101.
- W. M. Schmidt, On normal numbers, Pacific J. Math. 10 (1960), 661-672.

Partly normal numbers

- Cantor measure:

$$\mu = \mathop{\ast}_{k=1}^{\infty} \left[\frac{1}{2}\delta(0) + \frac{1}{2}\delta(2/3^k) \right].$$

$$\widehat{\mu}(\xi) = e^{-\pi i \xi} \prod_{j=1}^{\infty} \cos(2\pi 3^{-j}\xi), \quad \xi \in \mathbb{R}.$$

$$\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \left| \widehat{\mu}\left(h(b^n - b^m) \right) \right| < \infty. \quad \text{(DEL theorem)}$$

Proof overview

Lemma

There exists absolutely constants $C, \delta > 0$ *such that, for any* $2 \le b \not\sim 3$ *and any* $N \in \mathbb{N}$,

$$\sum_{n=0}^{N-1} |\widehat{\mu}(hb^n)| = \sum_{n=0}^{N-1} \prod_{j=1}^{\infty} |\cos(2\pi 3^{-j} hb^n)| < CN^{1-\delta}, \quad h \in \mathbb{Z} \setminus \{0\}.$$

DEL theorem:

$$\sum_{N=1}^{\infty} \frac{1}{N^2} \sum_{n=0}^{N-1} |\widehat{\mu}\,(hb^n)| < \infty.$$

Proof overview

Lemma (Cassels & Schmidt)

Let $2 \le r \not\sim 3$. *For any* $k \in \mathbb{N}$ *and* $n \in \{0, 1, \dots, 3^k - 1\}$, *let*

$$r^n = \lambda_0(n)3^0 + \lambda_1(n)3^1 + \cdots + \lambda_i(n)3^i + \cdots, \quad \lambda_i(n) \in \{0, 1, 2\},$$

be the 3*-ary expansion of* r^n. *Then* $(\lambda_l(n), \lambda_{l+1}(n), \dots, \lambda_{l+k-1}(n))$ *take precisely once every one of the* 3^k *possible values of* $\{0, 1, 2\}^k$, *as* n *runs from* 0 *to* $3^k - 1$, *where* $l \in \mathbb{N} \cup \{0\}$ *is such that* $r \equiv 1 \pmod{3^l}$ *and* $r \not\equiv 1 \pmod{3^{l+1}}$.

Normality & non-normality to restricted bases

- For $\mathcal{B}, \mathcal{B}' \subset \mathbb{N}\backslash\{1\}$, $\mathcal{B}$ and $\mathcal{B}'$ are called compatible if $\forall (b, b') \in \mathcal{B}\times\mathcal{B}'$, $b \nsim b'$.
- For any compatible $\mathcal{B}, \mathcal{B}'$, define

$$\mathcal{N}(\mathcal{B}, \mathcal{B}') := \{x \in [0,1] : \ x \text{ is normal to } \forall b \in \mathcal{B}, \\ \text{but not normal to } \forall b' \in \mathcal{B}'\}.$$

- For any compatible $\mathcal{B} \cup \mathcal{B}' = \mathbb{N} \setminus \{1\}$, Schmidt constructed a Cantor-like set $\mathcal{E} \subset \mathcal{N}(\mathcal{B}, \mathcal{B}')$. Pollington showed that

$$\dim_{\mathrm{H}} \mathcal{E} = 1.$$

Normality & non-normality to restricted bases

- A. D. Pollington, The Hausdorff dimension of a set of normal numbers, Pacific J. Math. 95 (1981), 193-204.
- W. M. Schmidt, Über die Normalität von Zahlen zu verschiedenen Basen, Acta Arith. 7 (1961/62), 299-309.

Theorem (Pramanik-Z.)

For every compatible pair $(\mathscr{B}, \mathscr{B}')$, $\mathscr{N}(\mathscr{B}, \mathscr{B}')$ *can support a Rajchman probability measure* μ.

- Rajchman measure: $\widehat{\mu}(\xi) \to 0$ as $|\xi| \to \infty$.

Set of uniqueness & multiplicity

Definition

A set $E \subset [0,1]$ is called a set of uniqueness, denoted by $E \in \mathcal{U}$, if any trigonometric series of the form

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx} = 0 \ \ \forall\, x \in [0,1] \setminus E \Longrightarrow c_n \equiv 0.$$

Otherwise, it is called a set of multiplicity.

- Originated from 1869, Riemann, Heine, Cantor .... It is well known that

$$\text{countable set} \subset \mathcal{U} \cap \text{L. measurable} \subset \text{L. null.}$$

Set of uniqueness & multiplicity

Theorem (Piatetcki-Shapiro, 1952; Kahane & Salem, 1963)

Let $E \subset [0,1]$ be a closed set. Then the following are equivalent:
(i) $\exists$ *a Rajchman probability measure* μ *supported on* E*;*
(ii) E *is a set of multiplicity.*

Kahane & Salem's question

Question (Kahane-Salem, 1964)

Can a set of non-normal numbers support a Rajchman measure?
(equivalently) Is any set of non-normal numbers a set of uniqueness?

- J. P. Kahane and R. Salem, Distribution modulo 1 and sets of uniqueness, Bull. Amer. Math. Soc. 70 (1964), 259-261.

- It was answered in affirmative by Lyons for the set of numbers which are not 2-normal.
  - R. Lyons, The measure of nonnormal sets, Invent. Math. 83 (1986), 605-616.

Super normal number

Definition (Z. Rudnick, 2002)

A real number $x = \sum_{k=1}^{\infty} a_k 2^{-k} \in [0,1]$ *is called 2-super normal (or Poisson generic) if*

$$\lim_{k\to\infty} \frac{1}{2^k} \sharp \left\{ \omega \in \{0,1\}^k : \ \sharp\{1 \le j \le 2^k : \ (a_j, a_{j+1}, \ldots, a_{j+k-1}) = \omega\} = i \right\}$$

$$=: \lim_{k\to\infty} Z_k(i;x) = \frac{e^{-1}}{i!}, \quad \forall \ i = 0, 1, 2, \ldots.$$

- Z. Rudnick & A. Zaharescu, The distribution of spacings between fractional parts of lacunary sequences, Forum Math. 14 (2002), 691-712.

Super normal number

- For $k = 3$, $2^k = 8$,
$$x = 0.10000100\cdots ,$$

(1) For $i = 0$, $Z_3(0; x) = 4/8$ (011, 101, 110, 111);
(2) For $i = 1$, $Z_3(1; x) = 2/8$ (001, 010);
(3) For $i = 2$, $Z_3(2; x) = 2/8$ (100, 000);
(4) For $i = 3$, $Z_3(1; x) = 0$.

Super normal number

Theorem (Peres-Weiss)

(i) *Lebesgue almost every* $x \in [0,1]$ *is 2-super normal.*

(ii) *If* $x \in [0,1]$ *is 2-super normal, then* x *is normal to* 2.

(ii) *The Champernowne number* $\xi_c := 0.1\,10\,11\,100\cdots$ *is not 2-super normal.*

- N. Alvarez, V. Becher & M. Mereb, Poisson generic sequences, Int. Math. Res. Not., (2023), 20970–20987.

Open problem

Question (Y. Perez & B. Weiss)

Let $C_{1/3}$ be the classical Cantor set and μ the Cantor-Lebesgue measure. Whether μ-a.e. $x \in C_{1/3}$ is super normal?

Thank you!