New Inequalities on Sobolev Type Spaces and Their Applications

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Outline

1. Affine Sobolev Inequalities

Embedding theorem for Sobolev spaces

- $1 \le p < n$ (Sobolev inequality):

$$\|f\|_{L^{\frac{np}{n-p}}} \le C_{n,p} \|\nabla f\|_{L^p}. \tag{S}$$

- $p = n$ (Moser–Trudinger inequality):

$$\sup_{f \in W^{1,n}, f \neq 0} \int_{\mathbb{R}^n} \Phi_n \left(n\omega_n^{\frac{1}{n-1}} \left[\frac{|f(x)|}{\|f\|_{L^n} + \|\nabla f\|_{L^n}} \right]^{n'} \right) dx < \infty, \tag{MT}$$

where $\Phi_n(t) := e^t - \sum_{k=0}^{n-1} \frac{t^k}{k!}$.

- $n < p < \infty$ (Morrey inequality):

$$\|f\|_{L^\infty} \le n^{-\frac{1}{p}} \omega_n^{-\frac{1}{n}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}} |\operatorname{supp} f|^{\frac{1}{n}-\frac{1}{p}} \|\nabla f\|_{L^p}. \tag{M}$$

Embedding theorem for Sobolev spaces

- They are only $SO(n)$ invariant: For any $f \in W^{1,p}$ and $\phi \in SO(n)$, $\|\nabla(f \circ \phi)\|_{L^p} = \|\nabla f\|_{L^p}$.
- Indeed, for any linear operator ϕ on $\mathbb{R}^n$, $\nabla(f \circ \phi) = \phi \cdot (\nabla f \circ \phi)$. Hence,

$$|\nabla(f \circ \phi)| = |\nabla f \circ \phi|$$

only if $\phi \in SO(n)$ (that is, orthogonal matrix with $\det \phi$=1).

Affine Sobolev inequality

Lutwak–Yang–Zhang's improvement:

$$\|f\|_{L^{\frac{np}{n-p}}} \le S_{n,p}\left|\Pi_p^* f\right|^{-\frac{1}{n}} \le C_{n,p}\|\nabla f\|_{L^p}. \tag{LYZ}$$

- The volume of polar projection body:

$$\left|\Pi_p^* f\right| := \frac{1}{n}\int_{\mathbb{S}^{n-1}} \left[\int_{\mathbb{R}^n} |\nabla_\xi f(x)|^p \, dx\right]^{-\frac{n}{p}} d\sigma(\xi).$$

- E. Lutwak, D. Yang and G. Zhang, J. Differential Geom. 62 (2002), 17–38.
- G. Zhang, J. Differential Geom. 53 (1999), 183–202.

Affine Sobolev inequality

- (LYZ) is $SL(n)$ invariant (affine).
- For any linear operator ϕ on $\mathbb{R}^n$ and any $\xi \in \mathbb{S}^{n-1}$,

$$\nabla_\xi (f \circ \phi) = \nabla (f \circ \phi) \cdot \xi = \langle \phi \cdot (\nabla f \circ \phi), \xi \rangle = \langle \nabla f \circ \phi, \phi^* \cdot \xi \rangle \, .$$

Therefore, by change of variables twice, we obtain, for any $\phi \in SL(n)$ (that is, $\det \phi = 1$),

$$\begin{aligned} \left| \Pi_p^* (f \circ \phi) \right| &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left[\int_{\mathbb{R}^n} |\langle \nabla f(\phi(x)), \phi^* \cdot \xi \rangle|^p \, dx \right]^{-\frac{n}{p}} d\sigma(\xi) \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \left[\int_{\mathbb{R}^n} |\nabla_\xi f(x)|^p \, dx \right]^{-\frac{n}{p}} d\sigma(\xi) = \left| \Pi_p^* f \right| . \end{aligned}$$

Proof of $|\Pi_p^* f|^{-\frac{1}{n}} \lesssim \|\nabla f\|_{L^p}$

Indeed, applying Hölder's inequality with $(\frac{n}{n+p})^{-1} + (-\frac{n}{p})^{-1} = 1$, we obtain

$$\int_{\mathbb{S}^{n-1}} \|\nabla_\xi f\|_{L^p}^p \, d\sigma(\xi) \geq \left[\int_{\mathbb{S}^{n-1}} d\sigma(\xi)\right]^{\frac{n+p}{n}} \left[\int_{\mathbb{S}^{n-1}} \|\nabla_\xi f\|_{L^p}^{-n} \, d\sigma(\xi)\right]^{-\frac{p}{n}}$$
$$= n\omega_n^{1+\frac{p}{n}} \left|\Pi_p^* f\right|^{-\frac{p}{n}}.$$

In addition, observe that

$$\int_{\mathbb{S}^{n-1}} \|\nabla_\xi f\|_{L^p}^p \, d\sigma(\xi) = \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |\nabla f(x) \cdot \xi|^p \, d\sigma(\xi) \, dx = \alpha_{n,p} \|\nabla f\|_{L^p}^p,$$

where $\alpha_{n,p} := \int_{\mathbb{S}^{n-1}} |e \cdot \xi|^p \, d\sigma(\xi)$ is a constant for any $e \in \mathbb{S}^{n-1}$. Hence $|\Pi_p^* f|^{-\frac{p}{n}} \leq \alpha_{n,p}^{\frac{1}{p}} (n\omega_n^{1+\frac{p}{n}})^{-\frac{1}{p}} \|\nabla f\|_{L^p}$. $\square$

Polar projection body: functional aspect

- For any $p \in [1, \infty)$, $f \in W^{1,p}$, and $\xi \in \mathbb{R}^n$, let

$$\|\xi\|_{\Pi_p^* f} := \left[\int_{\mathbb{R}^n} |\nabla f(x) \cdot \xi|^p \, dx \right]^{\frac{1}{p}}.$$

  The Polar projection body $\Pi_p^* f$ is defined to be the unit ball of $\| \cdot \|_{\Pi_p^* f}$.

- By the polar coordinate formula,

$$\left| \Pi_p^* f \right| = \int_{\|x\|_{\Pi_p^* f} \leq 1} dx = \int_{\mathbb{S}^{n-1}} \int_{r\|\xi\|_{\Pi_p^* f} \leq 1} r^{n-1} \, dr \, d\sigma(\xi)$$

$$= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \|\xi\|_{\Pi_p^* f}^{-n} \, d\sigma(\xi).$$

Polar projection body: geometric aspect

Let $p \in [1, \infty)$ and K be a convex body. Then

$$K \Longrightarrow \Pi_p K \text{ (projection body)} \Longrightarrow \Pi_p^* K \text{ (polar projection body)}.$$

Lutwak–Yang–Zhang:

- If $p = 1$ and $f := \mathbf{1}_K$,

$$\Pi_1^* f = \Pi_1^* K;$$

- If $p > 1$ and $f := (1 + \|x\|_K^{\frac{p}{p-1}})^{1-\frac{n}{p}}$ (where $\|\cdot\|_K$ is the Minkowski functional of K),

$$\Pi_p^* f = \widetilde{c}_{n,p} \Pi_p^* K.$$

Petty projection inequality

Let $p \in [1, \infty)$. Then, for any convex body K in $\mathbb{R}^n$,

$$|K|^{\frac{n-p}{n}} \left|\Pi_p^* K\right|^{\frac{p}{n}} \le \frac{\omega_n \omega_{p-1}}{\omega_{n+p-2}}. \tag{P}$$

Lutwak–Yang–Zhang: If $1 \le p < n$,

(P) (Petty projection inequality) $\Longleftrightarrow$ (LYZ) (L^p affine Sobolev inequality).

A well-known result:

Isoperimetric inequality $\Longleftrightarrow$ L^1 Sobolev inequality.

Petty projection inequality is also called affine isoperimetric inequality.

Applications

Applying affine isoperimetric inequality, Lutwak, Yang and Zhang in [Ann. Probab. 32 (2004), 757–774] showed:

Let $p \in [1, \infty)$ and $\lambda \in (\frac{n}{n+p}, \infty)$. If X and Y are independent random vectors in $\mathbb{R}^n$ with finite p-th moments, then

$$\mathbb{E}\left(|X \cdot Y|^p\right) \geq c_0 [N_\lambda(X) N_\lambda(Y)]^{p/n},$$

where c_0 is the best possible constant and N_λ is the λ-Rényi entropy of X.

More applications: Lutwak, Lv, Yang and Zhang [IEEE Trans. Inform. Theory 58 (2012), 1319–1327] and [IEEE Trans. Inform. Theory 59 (2013), 5592–5599].

Functional & geometric inequalities

- Brunn–Minkowski inequality: For any convex bodies K and L,

$$|K+L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}. \tag{BM}$$

- Minkowski inequality: For any compact E and any convex body K,

$$V_1(E,K) := \frac{1}{n} \liminf_{\varepsilon \to 0^+} \frac{|E+\varepsilon K| - |E|}{\varepsilon} \geq |E|^{\frac{1}{n'}} |K|^{\frac{1}{n}}. \tag{Mix}$$

- (BM)$\Longrightarrow$(Mix)$\Longrightarrow$ isoperimetric inequality. For the second implication: if let $E := B(\mathbf{0},1)$, then $V_1(E,K) = \frac{1}{n}|\partial E|$.

- A. Figalli, Quantitative stability results for the Brunn–Minkowski inequality, in: Proceedings of the International Congress of Mathematicians-Seoul 2014, Vol. III, 237–256, Kyung Moon Sa, Seoul, 2014.

Affine Moser–Trudinger–Morry inequalities

For any $p \in [1, \infty)$, if let $c_{n,p} := \left(\frac{n\omega_n\omega_{p-1}}{2\omega_{n+p-2}}\right)^{\frac{1}{p}}\omega_n^{\frac{1}{n}}$, then

- $p = n$:

$$\sup_{f \in W^{1,n}, f \neq \mathbf{0}} \int_{\mathbb{R}^n} \Phi_n \left(n\omega_n^{\frac{1}{n-1}} \left[\frac{|f(x)|}{\|f\|_{L^n} + c_{n,n}|\Pi_n^* f|^{-\frac{1}{n}}} \right]^{n'} \right) dx < \infty;$$

- $n < p < \infty$:

$$\|f\|_{L^\infty} \leq n^{-\frac{1}{p}}\omega_n^{-\frac{1}{n}} \left(\frac{p-1}{p-n} \right)^{1-\frac{1}{p}} c_{n,p} |\operatorname{supp} f|^{\frac{1}{n}-\frac{1}{p}} \left| \Pi_p^* f \right|^{-\frac{1}{n}}.$$

- Stronger than the classical ones (MT) and (M).

- A. Cianchi, E. Lutwak, D. Yang and G. Zhang, Calc. Var. Partial Differential Equations 36 (2009), 419–436.

Tool: Symmetrization inequality

- Let $E \subset \mathbb{R}^n$ be a Borel set of finite volume. The Schwarz symmetral $E^\star$ of E is defined to be the closed ball centered at $\mathbf{0}$ satisfying $|E^\star| = |E|$.
- Let $f : \mathbb{R}^n \to [0, \infty)$ be a measurable function. The Schwarz symmetral $f^\star$ of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$f^\star(x) := f^*(\omega_n |x|^n),$$

  where f^* is the nonincreasing rearrangement of f.
- If let $f := \mathbf{1}_E$, then $f^\star = \mathbf{1}_{E^\star}$.

Tool: Symmetrization inequality

- If let $\mu_f(t) := |\{x \in \mathbb{R}^n : |f(x)| > t\}|$, then
$$\mu_f = \mu_{f^*} = \mu_{f^\star}$$
and hence, for any $p \in [1, \infty]$,
$$\|f\|_{L^p} = \|f^\star\|_{L^p}.$$
- Pólya–Szegö inequality:
$$\|\nabla f^\star\|_{L^p} \le \|\nabla f\|_{L^p}.$$
- Affine Pólya–Szegö inequality (LYZ02 for $p \in [1, n)$ and CLYZ09 for $p \in [1, \infty)$):
$$\left|\Pi_p^* f^\star\right|^{-\frac{1}{n}} \le \left|\Pi_p^* f\right|^{-\frac{1}{n}}.$$

Idea of proofs (1/3)

Assume $f \in C_c^\infty$. By the coarea formula and Hölder's inequality with $(\frac{1}{p})^{-1} + (\frac{1}{1-p})^{-1} = 1$, we obtain

$$
\begin{aligned}
\|\nabla f\|_{L^p}^p &= \int_0^\infty \int_{|f(x)|=t} |\nabla f(x)|^{p-1} \, d\mathcal{H}^{n-1}(x)\, dt \\
&\geq \left[\int_{|f(x)|=t} d\mathcal{H}^{n-1}(x) \right]^p \left[\int_{|f(x)|=t} \frac{1}{|\nabla f(x)|} \, d\mathcal{H}^{n-1}(x) \right]^{1-p}.
\end{aligned}
$$

From the isoperimetric inequality, it holds that, for any $t \in (0, \infty)$,

$$
\begin{aligned}
\int_{|f(x)|=t} d\mathcal{H}^{n-1}(x) &= |\partial\{x \in \mathbb{R}^n : |f(x)| > t\}| \geq n\omega_n^{\frac{1}{n}} |\{x \in \mathbb{R}^n : |f(x)| > t\}| \\
&= n\omega_n^{\frac{1}{n}} \mu_f(t).
\end{aligned}
$$

Idea of proofs (2/3)

In addition, the following well-know fact holds: For almost every $t \in (0, \infty)$,

$$-\mu_f'(t) \geq \int_{|f(x)|=t} \frac{1}{|\nabla f(x)|} \, d\mathcal{H}^{n-1}(t).$$

Hence, by the definition of $f^\star$ and direct calculation,

$$\|\nabla f\|_{L^p}^p \geq n^p \omega_n^{\frac{p}{n}} \int_0^\infty \frac{\mu_f(t)^{p-\frac{p}{n}}}{-\mu_f'(t)^{p-1}} \, dt = \|\nabla f^\star\|_{L^p}^p.$$

This finishes the proof of the Pólya–Szegö inequality. $\square$

Idea of proofs (3/3)

For the proof of affine Pólya–Szegö inequality: Using the almost same idea and:

- isoperimetric inequality $\overset{\text{replaced by}}{\Longrightarrow}$ Petty projection inequality;
- But $\{x \in \mathbb{R}^n : |f(x)| > t\}$ may not be a convex body. To this end, Cianchi, Lutwak, Yang and Zhang used the Brunn–Minkowski theory to construct corresponding convex body.

- E. Lutwak, D. Yang and G. Zhang, Trans. Amer. Math. Soc. 356 (2004), 4359–4370.

PS inequality $\Longrightarrow$ affine Sobolev inequality

By direct calculation, for any f, $\|\nabla f^\star\|_{L^p} = c_{n,p}|\Pi_p^* f^\star|^{-\frac{1}{n}}$. Thus, applying the classical Sobolev inequality and the affine Pólya–Szegö inequality, we obtain

$$\begin{aligned}\|f\|_{L^{\frac{np}{n-p}}} &= \|f^\star\|_{L^{\frac{np}{n-p}}} \le C_{n,p}\|\nabla f^\star\|_{L^p} \\ &= C_{n,p}c_{n,p}\left|\Pi_p^* f^\star\right|^{-\frac{1}{n}} \le C_{n,p}c_{n,p}\left|\Pi_p^* f\right|^{-\frac{1}{n}}.\end{aligned}$$

$\square$

2. Affine Fractional Sobolev Inequalities

Fractional Sobolev inequality

Let $p \in [1, \infty)$ and $s \in (0, 1)$.

- Fractional Sobolev seminorm:

$$[f]_{W^{s,p}} := \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right]^{\frac{1}{p}}.$$

- If $s \in (0, 1)$ and $p \in [1, \frac{n}{s})$, then

$$\|f\|^p_{L^{\frac{np}{n-sp}}} \leq \widetilde{C}_{n,p} \frac{s(1-s)}{(n-sp)^{p-1}} [f]^p_{W^{s,p}}.$$

Fractional polar projection body

- For any $f \in W^{s,p}$ and $\xi \in \mathbb{R}^n$, the fractional L^p polar projection body $\Pi_p^{*,s} f$ is defined to be the unit ball in terms of

$$\|\xi\|_{\Pi_p^{*,s} f} := \left[\int_0^\infty t^{-sp} \int_{\mathbb{R}^n} |f(x+t\xi) - f(x)|^p \, dx \, \frac{dt}{t} \right]^{\frac{1}{sp}}.$$

- J. Haddad and M. Ludwig, Math. Ann. 388 (2024), 1091–1115.
- J. Haddad and M. Ludwig, J. Differential Geom. 129 (2025), 695–724.

Fractional polar projection body

- $SL(n)$ invariance: For any $\phi \in SL(n)$,

$$\Pi_p^{*,s}(f \circ \phi) = \phi^{-1}\Pi_p^{*,s} f.$$

- The polar coordinate implies

$$\left|\Pi_p^{*,s} f\right| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \|\xi\|_{\Pi_p^{*,s} f}^{-n}\, d\sigma(\xi).$$

Hölder's inequality gives $|\Pi_p^{*,s} f|^{-\frac{s}{n}} \lesssim [f]_{W^{s,p}}$.

Fractional polar projection body

- Recall that the famous Bourgain–Brezis–Mironescu (BBM) formula: For any $f \in W^{1,p}$,

$$\lim_{s \to 1-} p(1-s)[f]^p_{W^{s,p}} = \alpha_{n,p} \|\nabla f\|^p_{L^p}.$$

- Haddad and Ludwig proved the affine BBM formula:

$$\lim_{s \to 1-} p(1-s) \left|\Pi_p^{*,s} f\right|^{-\frac{sp}{n}} = \left|\Pi_p^* f\right|^{-\frac{p}{n}}.$$

- J. Bourgain, H. Brezis and P. Mironescu, Another look at Sobolev spaces, in: Optimal Control and Partial Differential Equations, IOS, Amsterdam, 2001, pp. 439–455.

Affine fractional Sobolev inequality

Haddad and Ludwig proved that, for any $s \in (0,1)$ and $p \in [1, \frac{n}{s})$,

$$\|f\|^p_{L^{\frac{np}{n-sp}}} \leq \widetilde{C}_{n,p} \frac{s(1-s)}{(n-sp)^{p-1}} n \omega_n^{\frac{n+sp}{n}} \left|\Pi_p^{*,s} f\right|^{-\frac{sp}{n}}.$$

- $SL(n)$ invariance.

In the sense of possibly non-optimal constant:

- Since $|\Pi_p^{*,s} f|^{-\frac{s}{n}} \lesssim [f]_{W^{s,p}}$, it is stronger than the fractional Sobolev inequality;
- Letting $s \to 1^-$, it goes back to (LYZ).

Idea of proofs: fractional PS inequality

For any convex body K, define

$$[f]_{W^{s,p}_K} := \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+sp}} \, dx \, dy \right]^{\frac{1}{p}},$$

where $\|\cdot\|_K$ is the Minkowski functional of K. Then the following Pólya–Szegö inequality holds:

$$[f^\star]_{W^{s,p}_{K^\star}} \le [f]_{W^{s,p}_K}. \tag{PS}$$

Idea: Writing $\|x-y\|_K^{-n-sp} = \int_0^\infty \mathbf{1}_{t^{-\frac{1}{n+sp}}K}(x-y)\, dt$ and using the Riesz rearrangement inequality:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(y)k(x-y)\, dx\, dy \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^\star(x)g^\star(y)k^\star(x-y)\, dx\, dy.$$

Fractional PS$\Longrightarrow$affine fractional PS (1/2)

(PS) implies the following:

$$\left|\Pi_p^{*,s} f^\star\right|^{-\frac{sp}{n}} \leq \left|\Pi_p^{*,s} f\right|^{-\frac{sp}{n}}.$$

Indeed, by the polar coordinate formula, we find that

$$[f]^p_{W_K^{s,p}} = \int_0^\infty \int_{\mathbb{S}^{n-1}} t^{-sp} \|\xi\|_K^{-n-sp} = \int_{\mathbb{S}^{n-1}} \|\xi\|_K^{-n-sp} \|\xi\|^{sp}_{\Pi_p^{*,s} f}\, d\xi.$$

Hence, from (PS), we infer that

$$\int_{\mathbb{S}^{n-1}} \|\xi\|_{K^\star}^{-n-sp} \|\xi\|^{sp}_{\Pi_p^{*,s} f^\star}\, d\xi \leq \int_{\mathbb{S}^{n-1}} \|\xi\|_K^{-n-sp} \|\xi\|^{sp}_{\Pi_p^{*,s} f}\, d\xi.$$

Fractional PS$\Longrightarrow$affine fractional PS (2/2)

Applying Hölder's inequality with $(\frac{n}{n+sp})^{-1} + (-\frac{n}{sp})^{-1} = 1$, we obtain

$$\begin{aligned}
&\frac{1}{n}\int_{\mathbb{S}^{n-1}} \|\xi\|_{K^\star}^{-n-sp}\|\xi\|_{\Pi_p^{*,s}f^\star}^{sp}\,d\xi \\
&\geq \left(\frac{1}{n}\int_{\mathbb{S}^{n-1}} \|\xi\|_{K^\star}^{-n}\right)^{1+\frac{sp}{n}} \left(\frac{1}{n}\int_{\mathbb{S}^{n-1}} \|\xi\|_{\Pi_p^{*,s}f^\star}^{-n}\,d\xi\right)^{-\frac{sp}{n}} \\
&= |K^\star|^{1+\frac{sp}{n}}|\Pi_p^{*,s}f^\star|^{-\frac{sp}{n}} = |K|^{1+\frac{sp}{n}}|\Pi_p^{*,s}f^\star|^{-\frac{sp}{n}}.
\end{aligned}$$

Letting $K := \Pi_p^{*,s}f$, we further have

$$|\Pi_p^{*,s}f|^{1+\frac{sp}{n}}|\Pi_p^{*,s}f^\star|^{-\frac{sp}{n}} \leq \frac{1}{n}\int_{\mathbb{S}^{n-1}} \|\xi\|_{\Pi_p^{*,s}f}^{-n}\,d\xi = |\Pi_p^{*,s}f|.$$

PS inequality$\Longrightarrow$affine Sobolev inequality

Similar to the first-order case: By direct calculation, for any f, $[f^\star]_{W^{s,p}} \sim |\Pi_p^{*,s} f^\star|^{-\frac{s}{n}}$. Thus, applying the classical fractional Sobolev inequality and the affine Pólya–Szegö inequality, we obtain

$$\begin{aligned} \|f\|_{L^{\frac{np}{n-sp}}} = \|f^\star\|_{L^{\frac{np}{n-sp}}} &\lesssim [f^\star]_{W^{s,p}} \\ &\sim \left|\Pi_p^{*,s} f^\star\right|^{-\frac{s}{n}} \leq \left|\Pi_p^{*,s} f\right|^{-\frac{s}{n}}. \end{aligned}$$

$\square$

3. Affine Fractional Moser–Trudinger–Morrey Inequalities

Fractional Morrey inequality

- Let $s \in (0,1)$ and $p \in (\frac{n}{s}, \infty)$. Then, for any $f \in W^{s,p}$ with $|\operatorname{supp} f| < \infty$,

$$\|f\|_{L^\infty} \lesssim [s(1-s)]^{\frac{1}{p}} |\operatorname{supp} f|^{\frac{s}{n}-\frac{1}{p}} [f]_{W^{s,p}}, \tag{1}$$

  where the implicit positive constant is independent of s.
- Letting $s \to 1-$ and applying the BBM formula, (1) goes back to (M) (in the sense of possibly non-optimal constant).

By the fractional Pólya–Szegö inequality, it is easy to establish the corresponding affine Morrey inequality:

(Domínguez–L.–Tikhonov–Yang–Yuan 2025)

Let $s \in (0,1)$ and $p \in (\frac{n}{s}, \infty)$.

(i) For any $f \in W^{s,p}$ with $|\operatorname{supp} f| < \infty$,

$$\|f\|_{L^\infty} \lesssim [s(1-s)]^{\frac{1}{p}} \, |\operatorname{supp} f|^{\frac{s}{n}-\frac{1}{p}} \left|\Pi_p^{*,s} f\right|^{-\frac{s}{n}}. \tag{2}$$

(ii) For any $p \in (n, \infty)$ and $f \in W^{1,p}$ with $|\operatorname{supp} f| < \infty$,

$$|\operatorname{supp} f|^{\frac{s}{n}-\frac{1}{p}} \left|\Pi_p^{*,s} f\right|^{-\frac{s}{n}} \le \frac{b_{n,p}^{1-s}}{[ps(1-s)]^{\frac{1}{p}}} \, |\operatorname{supp} f|^{\frac{1}{n}-\frac{1}{p}} \left|\Pi_p^{*} f\right|^{-\frac{1}{n}}.$$

Affine Morrey inequality

In the sense of possibly non-optimal constant:

- $|\Pi_p^{*,s} f|^{-\frac{s}{n}} \lesssim [f]_{W^{s,p}}$ implies that (2) is stronger than the fractional Morrey inequality (1);
- (ii) implies that (2) is stronger than the affine Morrey inequality of CLYZ;
- Combining (2) and the affine BBM formula, we go back to the affine Morrey inequality of CLYZ.

Fractional Moser–Trudinger Inequality

- For any $p \in [1, \infty)$ and any measurable f, let $\omega(f,t)_p := (\frac{1}{t^n \omega_n} \int_{|h|<t} \|\Delta_h f\|_{L^p}^p \, dh)^{\frac{1}{p}}$. Moreover, if $q \in [1, \infty]$ and $s \in (0,1)$, then let

$$[f]_{B^s_{p,q}} := \left\{ \int_0^\infty t^{-sq} \omega(f,t)_p^q \, \frac{dt}{t} \right\}^{\frac{1}{q}} < \infty.$$

- If $q \in [1, \infty)$ and $p \in (n, \infty)$, then there exists $\alpha_{p,q,n} \in (0, \infty)$, depending only on p, q, and n, such that

$$\sup_{f \in B^{\frac{n}{p}}_{p,q}, f \neq \mathbf{0}} \int_{\mathbb{R}^n} \Phi_{p,q}\left(\alpha_{p,q,n} \left[\frac{|f(x)|}{\|f\|_{L^p} + [f]_{B^{\frac{n}{p}}_{p,q}}} \right]^{q'} \right) dx < \infty, \qquad (3)$$

where $\Phi_{p,q}(t) := e^t - \sum_{k=0}^{\lceil p/q' \rceil - 1} \frac{t^k}{k!}$.

Fractional Moser–Trudinger Inequality

- The asymptotic behavior of $\alpha_{p,q,n}$ in (3) is still unknown.
- If we want to establish an affine variant of (3) which is pointwise stronger than the result of CLYZ, we need to retain $q' := n'$, hence

$$q = n \Longrightarrow B^{\frac{n}{p}}_{p,n}.$$

Besov polar projection body

Let $p, q \in [1, \infty)$ and $s \in (0, 1)$.

- For any $f \in B^s_{p,q}$ and $\xi \in \mathbb{R}^n$, the Besov polar projection body $\Pi^{*,s}_{p,q} f$ is defined to be the unit ball in terms of

$$\|\xi\|_{\Pi^{*,s}_{p,q} f} := \left[\int_0^\infty t^{-sq} \left[\int_{\mathbb{R}^n} |f(x + t\xi) - f(x)|^p \, dx \right]^{\frac{q}{p}} \frac{dt}{t} \right]^{\frac{1}{sq}}.$$

- O. Domínguez, Oscar, Y. Li, S. Tikhonov, D. Yang and W. Yuan, Math. Ann. 392 (2025), 3319–3366.

Besov polar projection body

- If $p = q$, then $\Pi^{*,s}_{p,p} f = \Pi^{*,s}_{p} f$.
- $SL(n)$ invariance: For any $\phi \in SL(n)$,

$$\Pi^{*,s}_{p,q}(f \circ \phi) = \phi^{-1} \Pi^{*,s}_{p,q} f.$$

- $|\Pi^{*,s}_{p,q} f|^{-\frac{s}{n}} \lesssim [f]_{B^s_{p,q}}$.

MT Inequality with optimal asymptotic factor

(Domínguez–L.–Tikhonov–Yang–Yuan 2025)

Let $n \geq 2$. Then there exists an absolute constant $c \in (0, \infty)$ such that, for any $\alpha \in (0, \frac{1}{n' c^{n'} e})$,

$$\sup_{p \in (n,\infty)} \sup_{f \in B^{\frac{n}{p}}_{p,n}, f \neq \mathbf{0}} \int_{\mathbb{R}^n} \Phi_{p,n}\left(\alpha \left\{ \frac{|f(x)|}{\|f\|_{L^p} + \Lambda_{n,p} (1 - \frac{n}{p})^{\frac{1}{p}} [f]_{B^{\frac{n}{p}}_{p,n}}} \right\}^{n'} \right) dx < \infty,$$

where $\Lambda_{n,p}$ depends only on n and p.

Main tools

This result is equivalent to the following embedding of Besov spaces: If $n \geq 2$, then there exists an absolute constant $c \in (0, \infty)$ such that, for any $p \in (n, \infty)$, $q \in [p, \infty)$, and $f \in B^{\frac{n}{p}}_{p,n}$,

$$\|f\|_{L^q} \leq c q^{\frac{1}{n'}} \left\{ \|f\|_{L^p} + \Lambda_{n,p} \left(1 - \frac{n}{p}\right)^{\frac{1}{p}} [f]_{B^{\frac{n}{p}}_{p,n}} \right\}. \tag{4}$$

- Using K-functionals and Littlewood–Paley characterizations to obtain the asymptotic factor.
- This asymptotic factor is optimal because we also obtain the following BBM type formula: If $f \in C^2_{\mathrm{c}}$, then

$$\lim_{p \to n+} \left(1 - \frac{n}{p}\right)^{\frac{1}{n}} [f]_{B^{\frac{n}{p}}_{p,n}} = \frac{\gamma_n}{n^{\frac{1}{n}}} \|\nabla f\|_{L^n}.$$

Relations with classical MT inequality

BBM type formula gives the following: For any $f \in C_c^2$,

$$\lim_{p\to n+} \int_{\mathbb{R}^n} \Phi_{p,n}\left(\alpha \left\{ \frac{|f(x)|}{\|f\|_{L^p} + \Lambda_{(n,p)}(1-\frac{n}{p})^{\frac{1}{p}}[f]_{B^{\frac{n}{p}}_{p,n}}} \right\}^{n'} \right) dx$$

$$= \int_{\mathbb{R}^n} \Phi_n\left(\alpha \left\{ \frac{|f(x)|}{\|f\|_{L^n} + \theta_n \gamma_n n^{-\frac{1}{n}} \|\nabla f\|_{L^n}} \right\}^{n'} \right) dx.$$

This means that, in the sense of possibly non-optimal constant, our MT inequality goes back to the classical one.

Relations with classical MT inequality

Moreover, the following holds:

(Domínguez–L.–Tikhonov–Yang–Yuan 2025)

Let $n \geq 2$. Then there exist positive constants $p_n \in (n, \infty)$ and $C_n \in (0, \infty)$, depending only on n, such that, for any $p \in (n, p_n)$ and $f \in \dot{W}^{1,n}$,

$$[f]_{B^{\frac{n}{p}}_{p,n}} \leq C_n \left(1 - \frac{n}{p}\right)^{-\frac{1}{n}} \|\nabla f\|_{L^n}.$$

Thus, in the sense of possibly non-optimal constant, our MT inequality is stronger than the classical one.

Affine fractional MT Inequality

(Domínguez–L.–Tikhonov–Yang–Yuan 2025)

Let $n \geq 2$. Then there exists an absolute constant $c \in (0, \infty)$ such that, for any $\alpha \in (0, \frac{1}{n' c^{n'} e})$,

$$\sup_{p\in(n,\infty)} \sup_{\substack{f\in B^{\frac{n}{p}}_{p,n} \\ f\neq \mathbf{0}}} \int_{\mathbb{R}^n} \Phi_{p,n}\left(\alpha\left[\frac{|f(x)|}{\|f\|_{L^p} + \Xi_{n,p}(1-\frac{n}{p})^{\frac{1}{p}}|\Pi^{*,\frac{n}{p}}_{p,n} f|^{-\frac{1}{p}}}\right]^{n'}\right) dx < \infty.$$

- Pólya–Szegö inequality for Besov polar projection bodies:

$$\left|\Pi_{p,q}^{*,s} f^{\star}\right|^{-\frac{s}{n}} \lesssim \left|\Pi_{p,q}^{*,s} f\right|^{-\frac{s}{n}}.$$

- This, together with (4), further implies

$$\|f\|_{L^q} \leq c q^{\frac{1}{n'}} \left\{ \|f\|_{L^p} + \Xi_{n,p} \left(1 - \frac{n}{p}\right)^{\frac{1}{p}} \left|\Pi_{p,n}^{*,\frac{n}{p}} f\right|^{-\frac{1}{p}} \right\}.$$

Relations with affine MT inequality

BBM type formula also holds for polar projection bodies: If $f \in C_c^2$, then

$$\lim_{p \to n+} \left(1 - \frac{n}{p}\right)^{\frac{1}{n}} \left|\Pi_{p,n}^{*,\frac{n}{p}} f\right|^{-\frac{1}{p}} = n^{-\frac{1}{n}} \left|\Pi_n^* f\right|^{-\frac{1}{n}}.$$

Hence, for any $f \in C_c^2$,

$$\lim_{p \to n+} \int_{\mathbb{R}^n} \Phi_{p,n}\left(\alpha\left[\frac{|f(x)|}{\|f\|_{L^p} + \Xi_{n,p}(1 - \frac{n}{p})^{\frac{1}{p}} |\Pi_{p,n}^{*,\frac{n}{p}} f|^{-\frac{1}{p}}}\right]^{n'}\right) dx$$

$$= \int_{\mathbb{R}^n} \Phi_n\left(\alpha\left[\frac{|f(x)|}{\|f\|_{L^n} + \theta_n(\frac{\omega_n}{2n})^{\frac{1}{n}} |\Pi_n^* f|^{-\frac{1}{n}}}\right]^{n'}\right) dx. \qquad (5)$$

Therefore, in the sense of possibly non-optimal constant, our affine MT inequality can go back to the classical one.

Relations with affine MT inequality

If $f \in W^{1,n}$, then there exist two positive constants $p_n \in (n, \infty)$ and $C_n \in (0, \infty)$, depending only on n, such that, for any $p \in (n, p_n)$,

$$\left(1 - \frac{n}{p}\right)^{\frac{1}{n}} \left|\Pi_{p,n}^{*,\frac{n}{p}} f\right|^{-\frac{1}{p}} \leq C_n 2(2n)^{\frac{1}{n}} \Gamma\left(1 + \frac{1}{n}\right) \omega_n^{-\frac{1}{p}} \left|\Pi_n^* f\right|^{-\frac{1}{n}}.$$

This also means that, in the sense of possibly non-optimal constant, our affine MT inequality is stronger than the classical one.

Thank you!