调和分析在几何测度论与压缩感知、机器学习中的一些应用

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杭州　2025年12月21日

一、几何测度论中的应用

- $x, x+d, x+2d$: 3AP- 3 term arithmetic progressions

$$R(N) := \sup\{\#(A): \ A \subset \{0,1,\dots,N-1\}, A \text{ contains no 3AP}\}.$$

Theorem

$$Ne^{-c(\log N)^{\frac{1}{2}}} \ll R(N) \ll Ne^{-c(\log N)^{\frac{1}{9}}}$$

- Upper bound:
  Roth (1953 JLMS): $\frac{N}{\log\log N}$;
  Heath-Brown (1987 JLMS); Szemerédi (1990 AMH); Bourgain (1999 GAFA); Sanders(2011 Annals); Bloom-Sisask (2020 Acta): $\frac{N}{(\log N)^c}$;
  Kelley-Meka(2023): $Ne^{-c(\log N)^{\frac{1}{9}}}$.
- Lower bound: Behrend (1946 PNAS); Elkin (2011 IJM), Green and Wolf

Related problems

- Different spaces, different scales, different structures
- Every subset of prime of positive upper density contains an arithmetic progression of length three. (Green, 2005, Ann.)
- The primes contain arbitrarily long arithmetic progressions. (Green and Tao, 2008, Ann.)
- Kakeya Conjecture: Every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension n.
- Falconer Distance Set Conjecture: If $E \subset \mathbb{R}^n$ has Hausdorff dimension lager than $n/2$, $E - E$ has positive Lebesgue measure.
- Other structures: $x, x+t, x+P(t)$; triangle; square; infinite set (Erdös)...

Continuous case

- If $E \subset [0,1]$ contains no 3-AP, then $|E| = 0$.
- How "big" is E with $|E| = 0$?
- Hausdorff measure $\mathcal{H}^s$ of A, $s \geq 0$, is defined as

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

where, for $\delta \in (0, \infty]$,

$$\mathcal{H}^s_\delta(A) := \inf \left\{ C(s) \sum_j d(E_j)^s : \ A \subset \cup_j E_j, d(E_j) < \delta \right\}.$$

The Hausdorff dimension of $A \subset \mathbb{R}^n$ is

$$\dim_{\mathbb{H}} A := \inf\{s : \ \mathcal{H}^s(A) = 0\} = \sup\{s : \ \mathcal{H}^s(A) = \infty\}.$$

Lemma (Frostman's Lemma)

For $A \subset \mathbb{R}^n$, $\mathcal{H}^s(A) > 0$ if and only if there is a probability μ support on A such that

$$\mu(B(x,r)) \leq r^s, \forall x \in \mathbb{R}^n, r > 0.$$

In particular,

$$\dim_{\mathbb{H}} A := \sup\{s \leq n : \ \exists \mu \in \mathcal{M}(A), \ s.t. \ \mu(B(x,r)) \leq r^s, \forall x \in \mathbb{R}^n\}.$$

- The s-energy of μ is defined as

$$I_s(\mu) := \iint |x-y|^{-s} \, d\mu(x) \, d\mu(y) = \gamma(n,s) \int |\widehat{\mu}(x)|^2 |x|^{s-n} \, dx.$$

- $\dim_{\mathbb{H}} A := \sup\{s \leq n : \ \exists \mu \in \mathcal{M}(A), \ s.t. \ I_s(\mu) < \infty\}.$

Fourier dimension and Random fractal

- The Fourier dimension of $A \subset \mathbb{R}^n$ is

$$\dim_{\mathbb{F}} A := \sup\{s \le n : \ \exists \mu \in \mathcal{M}(A)\ s.t. |\widehat{\mu}(x)| \le |x|^{-s/2}, \forall x \in \mathbb{R}^n\}.$$

- Example: $|\widehat{\delta}(x)| = 1$, $|\widehat{\mathbb{1}}_{[0,1]}(x)| = |\int_0^1 e^{-2\pi ixt} dt| \lesssim |x|^{-1}$,
- If $|\widehat{\mu}(x)| \le |x|^{-s/2}$, then $I_t(\mu) < \infty$, $\forall t < s$. Thus,

$$\dim_{\mathbb{F}}(A) \le \dim_{\mathbb{H}}(A).$$

- A set A is a Salem set, if $\dim_{\mathbb{F}}(A) = \dim_{\mathbb{H}}(A)$.
- Cantor set C, $\dim_{\mathbb{H}} C = \frac{\log 2}{\log 3}$ and $\dim_{\mathbb{F}} C = 0$.
- Random Cantor set C, $\dim_{\mathbb{H}} C = \dim_{\mathbb{F}} C = \frac{\log 2}{\log 3}$.

Discrete to Continuous

- Keleti construct a compact subset E of $[0, 1]$ with Hausdorff dimension 1 such that E contains no 3-term arithmetic progressions.
- Shmerkin proved that there exist a compact Salem set E with Fourier dimension 1 such that E contains no 3-term arithmetic progressions.

- T. Keleti, A 1-dimensional subset of the reals that intersects each of its translates in at most a single point, Real Anal. Exchange 24, no. 2 (1998/1999): 843-4.
- P. Shmerkin, Salem sets with no arithmetic progressions, Int. Math. Res. Not. IMRN 2017, no. 7, 1929-1941.

Structural richness of Random fractal

$$\mathcal{F} := \{ax + by = cz : \ a, b, c \in \mathbb{Z}_+ \text{ and } a + b = c\}$$

- Fraser and Pramanik show that there exist a compact set with Hausdorff dimension 1 such that E contains no solutions of any equation in $\mathcal{F}$.
- R. Fraser and M. Pramanik, Large Sets Avoiding Patterns, Anal. PDE 11 (2018), no. 5, 1083-1111.

Structural richness of Random fractal

$$\mathcal{F} := \{ax + by = cz : \ a, b, c \in \mathbb{Z}_+ \text{ and } a + b = c\}$$

Theorem (L. and Pramanik 2022)

If $\dim_{\mathbb{F}} E > \frac{2}{3}$, then $\exists a, b, c \in \mathbb{Z}_+$ and distinct $x, y, z \in E$ such that

$$ax + by = cz.$$

Theorem (L. and Pramanik 2022)

For any finite set $\mathcal{F}_0 \subset \mathcal{F}$, $\exists E \subset [0, 1]$ with $\dim_{\mathbb{F}} E = 1$ such that E contains no nontrivial solution of $f = 0$, for any $f \in \mathcal{F}_0$.

- Y. Liang and M. Pramanik, Fourier dimension and avoidance of linear patterns, Adv. Math. 399 (2022), Paper No. 108252.

Structural richness of Random fractal

Theorem (L. and Pramanik 2022)

Given $v \in \mathbb{N}$*,* $v \geq 2$*, let* $E \subseteq [0,1]$ *be a closed set satisfying* $\dim_{\mathbb{F}}(E) > \frac{2}{v+1}$*; i.e., there exist some* $\beta > \frac{1}{v+1}$*, a probalility measure* μ *supported on* E *and some positive constant* C *such that*

$$|\hat{\mu}(\xi)| \leq C(1+|\xi|)^{-\beta}.$$

Then there exists $\{m_0, \ldots, m_v\} \subset \mathbb{N}$ *satisfying* $m_0 = \sum_{i=1}^{v} m_i$*, such that* E *contains a nontrivial solution of the equation*

$$\sum_{i=1}^{v} m_i x_i = m_0 x_0.$$

Structural richness of Random fractal

- Sidon set: For any $\{x, y, z, w\} \subset E$ with $x < y \leq z < w$, $x - y \neq z - w$.
- Keleti construct a compact Sidon set E of $[0, 1]$ with Hausdorff dimension 1.
- T. Keleti, A 1-dimensional subset of the reals that intersects each of its translates in at most a single point, Real Anal. Exchange 24, no. 2 (1998/1999): 843-4.

Theorem (L. and Pramanik)

For any compact set $E \subset [0, 1]$ with $\dim_{\mathbb{F}}(E) > 1/2$, E contains a nontrivial solution of $x - y = z - w$.

On the other side, there exists a Sidon set $E \subset [0, 1]$ with $\dim_{\mathbb{F}}(E) \geq \frac{\sqrt{2}}{4+3\sqrt{2}}$, i. e. E contains no nontrivial solution of $x + y = z + w$.

调和分析的应用

- 基本概念- Fourier 维数
- 证明任意性：$\mathbf{t} := t_1, t_2$ 且$t_1 + t_2 = 1$,

$$\langle \Lambda_{\mathbf{t}}, \mathfrak{f} \rangle := \int_{[0,1]^2} \mathfrak{f}(x_1, x_2)\, \mu\Big(t_1 x_1 + t_2 x_2\Big)\, d\mu(x_1) d\mu(x_2).$$

- KEY STEP: 证明

$$F^{[\mathfrak{f}]}(\mathbf{t}) := \langle \Lambda_{\mathbf{t}}, \mathfrak{f} \rangle$$
$$= \int_{\mathbb{R}^3} \widehat{\mu}(\xi)\widehat{\mu}(\eta_1)\widehat{\mu}(\eta_2)\widehat{\mathfrak{f}}\,(-\eta_1 - t_1\xi, -\eta_2 - t_2\xi)\; d\xi\, d\eta_1\, d\eta_2$$

在$(0,1)^2$ 连续.

- 证明存在性：使用概率Bernstein 不等式

$$\mathbb{P}\left(\left|\widehat{\mu_{n+1}}(k) - \widehat{\mu_n}(k)\right| \geq Q_{n+1}^{-\sigma/2}\right) \lesssim 4\exp\left(-Q_n^{-\sigma}\right).$$

二、在压缩感知中的应用

整体框架

- 问题：已知未知信号$x \in \mathbb{R}^n$ 稀疏，通过硬件可设置测量矩阵$A \in \mathbb{R}^{n\times m}$, 得到观测值$y = Ax \in \mathbb{R}^m$. 即假设$\#supp(x) = s \ll m \ll n$, 求解$Ax = y$.
- 模型：凸松弛，将NP-hard 问题

$$x = argmin\{\|z\|_0 : Az = y\}$$

转化为凸优化问题

$$x = argmin\{\|z\|_1 : Az = y\}$$

- 理论问题：

1) A 需要满足的条件
2) 存在性与具体构造
3) s, m, n之间的关系

- 算法：基追踪、软阈值、ADMM
- 应用：医学成像MRI、图片恢复、低秩矩阵恢复、相位恢复等

主要结果

1) A 需要满足的条件:
Tao 等人提出RIP条件：$\forall$ s-稀疏向量$x \in \mathbb{R}^n$,

$$(1-\delta)\|x\|^2 \leq \|Ax\|^2 \leq (1+\delta)\|x\|^2.$$

当$\delta_{2s} < \frac{1}{\sqrt{2}}$时，可以通过$\ell_1$极小精确恢复.

2) 存在性与具体构造:
在$m \gtrsim s\ln(N/s)$条件下时, 高斯随机矩阵满足$\delta_{2s} < \frac{1}{\sqrt{2}}$的RIP 条件.

- 其他模型：$\ell_1 - \ell_2$, $\ell_p, p < 1$, 加权ℓ_1等
- 稳定性、鲁棒性等：ρ-NSP条件、非高斯噪声等

调和分析的应用与进一步的问题

- 已知的应用

1) 许多信号本身不具有稀疏性，是在某种基展开下具有稀疏性。这里用的Fourier 变换、小波等展开方式.
2) 随机矩阵的构造中，有一类就是Fourier 矩阵的随机选取.

- 进一步的可能

1) 调和分析中的函数空间理论（如Besov空间）用于刻画信号的稀疏性程度，从而分析压缩感知恢复误差的上下界.
2) 调和分析中的多尺度几何分析（如脊波、曲波）能够处理高维信号（如图像、视频）中的边缘和纹理结构，推广了压缩感知对结构化稀疏信号的应用.

三、在机器学习中的应用

整体框架

- 问题：已知$(x_1, y_1), \cdots, (x_n, y_n)$, 求映射关系$f : x \to y$.
- 模型：$f(x) = f_1 \circ f_2 \circ \cdots \circ f_N(x)$, 其中$f_i(x) = \sigma(A_i x + b_i)$, σ 为一非线性激活函数, 如ReLu等.
- 问题：

1) 模型的具体架构: 全链接, CNN, Transformer等
2) f 的条件与逼近误差、泛化误差、优化误差等
3) 可解释性

- 算法：梯度下降法
- 实际应用：调参

调和分析的应用与进一步的问题

- 已知的应用

1) Fourier 神经算子，通过傅里叶变换、小波变换将信号转换到频率域
2) 假定f 属于再生核希尔伯特空间（RKHS），得到逼近误差

- 进一步的问题

1) 机器学习模型（如神经网络）可以看作是在特定函数空间中寻找函数。调和分析定义了这些空间的范数（如Sobolev范数），用于度量函数的“复杂度”或“平滑度”，这与模型的泛化能力直接相关。
2) 将处理规则信号（如音频）的工具扩展到处理非欧数据（如图、流形）。

Thank you ！