Heat kernel-based p-energy norms on metric measure spaces

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Outline

- Historical background and motivation
- Preliminaries
- Results on metric measure spaces
- Results on fractals

History

In 2001, Bourgain, Brezis and Mironescu proposed the convergence results in [BBM01]:

$$\lim_{\sigma \uparrow 1}(1-\sigma)\int_D\int_D \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\sigma}}dxdy = C_{n,p}\int_D |\nabla u(x)|^p dx, \qquad (1)$$

where D is a smooth area in $\mathbb{R}^n$, which states that multiplying by a scaling factor $1-\sigma$, the fractional Gagliardo semi-norm of a function converges to the first-order Sobolev semi-norm as $\sigma \to 1$.

History

- The term '$\int_D |\nabla u(x)|^p \ \mathrm{d}x$' can be regarded as a local energy form, for which the corresponding generator is the *p-Laplacian*

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u).$$

- The term '$\int_D \int_D |u(x)-u(y)|^p |x-y|^{-d-p\sigma} dxdy$' can be regarded as a non-local energy form with kernel $|x-y|^{-d-p\sigma}$, which is associated with *fractional p-Laplacian operator* $(-\Delta_p)^\sigma$ in $\mathbb{R}^d$.

Hence, (1) builds up a relationship between local energy forms and a non-local energy form.

p-energy form or norm

- $p = 2$: Dirichlet form
- $p \neq 2$: p-energy form

History for $p = 2$

For other metric spaces, a natural analogue to BBM convergence is the convergence of Besov semi-norms $(1-\sigma)B^{\sigma}_{2,2}$ to $B^{\sigma^*}_{2,\infty}$, where σ^* is a fixed number (usually termed as 'walk dimension') in

- Sierpiński gasket [PP08,Yang18]
- Sierpiński carpet [GY19]
- p.c.f. self-similar set [GL20A,GL20T]
- metric measure space[Yang22]

Open problem

Gu and Lau left two open problems in [GL20A,GL20T]

- consider the convergence of $B^{\sigma}_{p,p}$-norms to the $B^{\sigma^*}_{p,\infty}$-norm see [GYZ22].
- extend such convergence to more general settings under more general ‘weak-monotonicity properties’?

More general settings?

Euclidean space, manifold, fractal or metric measure space.

History of p-energy norms

- Herman, Peirone and Strichartz proposed a notion of p-energy defined on the Sierpiński gasket for any $p \in (1, \infty)$ in [HPS04].
- Hu, Ji and Wen took the Sierpiński gasket in $\mathbb{R}^2$ as an example and and discussed the Hajłasz-Sobolev type space as it is related to the p-energy see [HJW07].
- Constructions of p-energy $(1 < p < \infty)$ are based on the graph-approximation to the underlying space, for p.c.f. self-similar sets by Cao, Gu and Qiu see [CGQ22], for the Sierpiński carpet see [Shimizu21], and for metric measure spaces [Kigami21].

History of p-energy norms

- Heat kernel-based p-energy norms basically appeared in [PP10].
- heat semigroup-based norms are systematically studied by Alonso Ruiz, Baudoin et al.for $1 \leq p < \infty$ in [ABCRST20J, ABCRST20C, ABCRST21], where they focus on generalising classical analysis results including Sobolev embeddings, isoperimetric inequalities and the class of bounded variation functions etc.
- Baudoin proved weak-monotonicity properties with p-energy norms in [Baudoin22] by p-Poincare inequality etc.

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Motivation

- to generalise the celebrated 'Bourgain-Brezis-Mironescu (BBM) convergence' (1) of p-energy forms $(1 < p < \infty)$ to the metric measure space
- to verify various weak-monotonicity properties

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Metric measure space

Let (M, d, μ) be a metric measure space. Assume that (M, d) has more than one point with positive diameter. Fix a value (which could be infinite)

$$R_0 \in (0, \mathrm{diam}(M)]$$

that will be used for localization throughout the paper, where

$$\mathrm{diam}(M) := \sup\{d(x, y) : x, y \in M\} \in (0, \infty]$$

is infinite if M is unbounded and is finite if M is bounded.

Metric measure space

The measure μ is assumed to be Borel regular with the following positivity and *volume doubling property*: there exists a constant $C_d > 0$ such that for every $x \in M$, $0 < r < \infty$,

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty, \tag{2}$$

where

$$B(x, r) := \{y \in M : d(x, y) < r\}$$

denotes the open metric ball centered at $x \in M$ with radius $r > 0$. Denote

$$V(x, r) := \mu(B(x, r)).$$

If (2) holds, then there exists $\alpha_1 > 0$ such that

$$\frac{V(x,R)}{V(x,r)} \leq C_d \left(\frac{R}{r}\right)^{\alpha_1}, \tag{3}$$

for all $x \in M$, $0 < r \leq R < \infty$.
We say that μ is α-*regular* if there exists a constant $C \geq 1$ such that for all $0 < r < R_0$,

$$C^{-1} r^{\alpha} \leq V(x,r) \leq C r^{\alpha}. \tag{4}$$

We remark that when μ is α-regular, then (3) holds with $\alpha_1 = \alpha$.

Heat kernel

Given a metric measure space (M, d, μ), a family $\{p_t\}_{t>0}$ of non-negative measurable functions $p_t(x, y) : M \times M \to \mathbb{R}_+$ on $M \times M$ is called a *heat kernel*, if the following conditions are satisfied for μ-almost all $x, y \in M$ and $s, t > 0$:

(1) Symmetry: $p_t(x, y) = p_t(y, x)$.

(2) The total mass inequality:

$$\int_M p_t(x, y) d\mu(y) \leq 1.$$

(3) Semigroup property:

$$p_{s+t}(x, y) = \int_M p_s(x, z) p_t(z, y) d\mu(z).$$

(4) Strong continuity: for any $f \in L^2(M, \mu)$,

$$\int_M p_t(x, y) f(y) d\mu(y) \to f(x) \quad \text{as} \quad t \downarrow 0, \text{strongly in} \quad L^2(M, \mu).$$

Two-sided heat kernel estimates

We say that a heat kernel $\{p_t\}_{t>0}$ satisfies the *two-sided* heat kernel estimates, if for all $t \in (0, R_0^{\beta^*})$ and μ-almost all $x, y \in M$:

$$\frac{c_1}{V(x, t^{1/\beta^*})} \exp\left(-c_2 \left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right) \le p_t(x,y)$$

$$\le \frac{c_3}{V(x, t^{1/\beta^*})} \exp\left(-c_4 \left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right), \tag{5}$$

where $\beta^* > 1$ is a fixed number.

Two-sided heat kernel estimates

Such estimates hold for many classical cases.

- The classical Gauss-Weierstrass function in $\mathbb{R}^n$

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right),$$

- Riemannian manifold with non-negative Ricci curvature

$$p_t(x,y) \asymp \frac{C}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{ct}\right),$$

- Fractal spaces

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta^*}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta^*}}\right)^{\frac{\beta^*}{\beta^*-1}}\right)$$

Definition of Dirichlet form

It is well-known that, a heat kernel $\{p_t\}_{t>0}$ can deduce a Dirichlet form in $L^2(M,\mu)$ given by

$$\mathcal{E}(u,u) = \lim_{t\to 0^+} \mathcal{E}_t(u,u), \;\; u \in L^2(M,\mu),$$
$$\mathcal{D}(\mathcal{E}) = \{u \in L^2(M,\mu) : \mathcal{E}(u,u) < \infty\},$$

where

$$\mathcal{E}_t(u,u) := \frac{1}{2t}\int_M\int_M |u(x)-u(y)|^2 p_t(x,y)d\mu(y)d\mu(x). \tag{6}$$

The limit exists, since for any given $u \in L^2(M,\mu)$, $t \to \mathcal{E}_t(u,u)$ is monotone decreasing by using the spectral theorem.

Definition of p-energy norms

We adapt the heat kernel-based norms to define p-energy norm in $L^p(M,\mu)$ for a heat kernel $\{p_t\}_{t>0}$ as

$$E^{\sigma}_{p,\infty}(u) := \sup_{t\in(0,R_0^{\beta^*})} \frac{1}{t^{p\sigma/\beta^*}} \int_M \int_M |u(x)-u(y)|^p p_t(x,y) d\mu(y) d\mu(x),$$

with its domain

$$\mathcal{D}(E^{\sigma}_{p,\infty}) := \{u \in L^p(M,\mu) : E^{\sigma}_{p,\infty}(u) < \infty\}.$$

It is a natural way to extend the Dirichlet form case for $p=2$ to $p\in(1,\infty)$ and we will call them heat kernel-based p-energy norms.

Definition of p-energy norms

To extend the BBM convergence from $p = 2$ to $1 < p < \infty$, it is quite natural to define

$$E^{\sigma}_{p,p}(u) := \int_0^{R_0^{\beta^*}} \frac{1}{t^{p\sigma/\beta^*}} \left(\int_M \int_M |u(x) - u(y)|^p p_t(x,y) d\mu(x) d\mu(y) \right) \frac{dt}{t},$$

with domain

$$\mathcal{D}(E^{\sigma}_{p,p}) := \{u \in L^p(M,\mu) : E^{\sigma}_{p,p}(u) < \infty\}.$$

We also consider the convergence of heat kernel-based p-energy norms $E^{\sigma}_{p,p}$ to $E^{\sigma_p^{\#}}_{p,\infty}$, but unfortunately $E^{\sigma}_{p,p}$ isn't monotone decreasing, so we need weak-monotonicity properties.

Definition of p-energy norms

For simplicity, for $u \in L^p(M,\mu)$ and $\sigma, t > 0$, denote $\Psi_u^\sigma(t)$ by

$$\Psi_u^\sigma(t) = \frac{1}{t^{p\sigma/\beta^*}} \int_M \int_M |u(x) - u(y)|^p p_t(x,y) d\mu(y) d\mu(x), \tag{7}$$

then

$$E_{p,\infty}^\sigma(u) = \sup_{t \in (0, R_0^{\beta^*})} \Psi_u^\sigma(t), \ \ E_{p,p}^\sigma(u) = \int_0^{R_0^{\beta^*}} \Psi_u^\sigma(t) \frac{dt}{t}. \tag{8}$$

weak-monotonicity properties(KE)

Definition 1

We say that a metric measure space (M, d, μ) with heat kernel $\{p_t\}_{t>0}$ satisfies property **(KE)** with $\sigma > 0$, if there exists a constant $C > 0$ such that for all $u \in \mathcal{D}(E^{\sigma}_{p,\infty})$,

$$\sup_{t\in(0,R_0^{\beta^*})} \Psi^{\sigma}_u(t) \leq C \liminf_{t\to 0} \Psi^{\sigma}_u(t),$$

where $\Psi^{\sigma}_u(t)$ is defined as in (9). We further say that property $(\widetilde{KE})$ is satisfied with $\sigma > 0$, if there exists a constant $C > 0$ such that for all $u \in \mathcal{D}(E^{\sigma}_{p,\infty})$,

$$\limsup_{t\to 0} \Psi^{\sigma}_u(t) \leq C \liminf_{t\to 0} \Psi^{\sigma}_u(t).$$

Definition of p-energy norms

For $\sigma > 0$, define semi-norms $[u]_{B^{\sigma}_{p,\infty}}$ for $u \in L^p(M,\mu)$ $(p > 1)$ by

$$[u]^p_{B^{\sigma}_{p,\infty}} := \sup_{r\in(0,R_0)} r^{-p\sigma} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x)-u(y)|^p d\mu(y) d\mu(x), \quad (9)$$

and define $B^{\sigma}_{p,\infty} := B^{\sigma}_{p,\infty}(M) = \{u \in L^p(M,\mu) : [u]_{B^{\sigma}_{p,\infty}} < \infty\}$.
Naturally, define semi-norms $[u]_{B^{\sigma}_{p,p}}$ by

$$[u]^p_{B^{\sigma}_{p,p}} := \int_0^{R_0} r^{-p\sigma} \left(\int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x) - u(y)|^p d\mu(y) d\mu(x) \right) \frac{dr}{r}, \quad (10)$$

and define $B^{\sigma}_{p,p} := B^{\sigma}_{p,p}(M) = \{u \in L^p(M,\mu) : [u]_{B^{\sigma}_{p,p}} < \infty\}$.

Some Conditions about p-energy norms

The space $B^{\sigma}_{p,\infty}$ coincides with the Korevaar-Schoen space $KS^{\sigma}_{p,\infty}$ in [Baudoin22] with the following different norm (switch 'sup' to 'limsup')

$$\|u\|^p_{KS^{\sigma}_{p,\infty}} = \limsup_{r\to 0} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} \frac{|u(x)-u(y)|^p}{r^{p\sigma}} d\mu(y)d\mu(x) < \infty. \tag{11}$$

Some Conditions about p-energy norms

In our paper, we define the *critical exponent of* (M, d, μ) by

$$\sigma_p^{\#} = \sup\{\sigma > 0 : B^{\sigma}_{p,\infty} \text{ contains non-constant functions}\}.$$

In many related studies the critical exponent is defined by

$$\sigma_p^{*} = \sup\{\sigma > 0 : B^{\sigma}_{p,\infty} \text{ is dense in } L^p(M, \mu)\}.$$

Clearly, $\sigma_p^{*} \leq \sigma_p^{\#}$.

Definition of p-energy norms

Similarly, for Besov norms, denote

$$\Phi_u^\sigma(r) = r^{-p\sigma} \int_M \frac{1}{V(x,r)} \int_{B(x,r)} |u(x) - u(y)|^p d\mu(y) d\mu(x), \qquad (12)$$

then

$$[u]^p_{B^\sigma_{p,\infty}} = \sup_{r\in(0,R_0)} \Phi_u^\sigma(r), \; [u]^p_{B^\sigma_{p,p}} = \int_0^{R_0} \Phi_u^\sigma(r) \frac{dr}{r},$$

$$\|u\|^p_{KS^\sigma_{p,\infty}} = \limsup_{r\to 0} \Phi_u^\sigma(r).$$

Weak-monotonicity properties(NE)

Definition 2

We say that a metric measure space (M, d, μ) satisfies property **(NE)** with $\sigma > 0$ if there exists $C > 0$ such that for all $u \in B^{\sigma}_{p,\infty}$,

$$\sup_{r \in (0,R_0)} \Phi^{\sigma}_{u}(r) \le C \liminf_{r \to 0} \Phi^{\sigma}_{u}(r). \tag{13}$$

In addition, we say that property $(\widetilde{NE})$ is satisfied with $\sigma > 0$, if there exists $C > 0$ such that for all $u \in B^{\sigma}_{p,\infty}$,

$$\limsup_{r \to 0} \Phi^{\sigma}_{u}(r) \le C \liminf_{r \to 0} \Phi^{\sigma}_{u}(r).$$

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Main Result

Theorem 3

Suppose that (M, d, μ) *with heat kernel* $\{p_t\}_{t>0}$ *satisfies property (KE) with* $\tilde{\sigma}_p > 0$. *Then there exists a positive constant* C *such that for all* $u \in B^{\tilde{\sigma}_p}_{p,\infty}$,

$$C^{-1} E^{\tilde{\sigma}_p}_{p,\infty}(u) \leq \liminf_{\sigma \uparrow \tilde{\sigma}_p} (\tilde{\sigma}_p - \sigma) E^{\sigma}_{p,p}(u) \leq \limsup_{\sigma \uparrow \tilde{\sigma}_p} (\tilde{\sigma}_p - \sigma) E^{\sigma}_{p,p}(u) \leq C E^{\tilde{\sigma}_p}_{p,\infty}(u). \quad (14)$$

Main Result

Theorem 4

Suppose that (M, d, μ) *satisfies property (NE), then there exists a positive constant* C *such that for all* $u \in B^{\sigma_p^\#}_{p,\infty}$,

$$C^{-1}[u]^p_{B^{\sigma_p^\#}_{p,\infty}} \leq \liminf_{\sigma \uparrow \sigma_p^\#}(\sigma_p^\# - \sigma)[u]^p_{B^\sigma_{p,p}} \leq \limsup_{\sigma \uparrow \sigma_p^\#}(\sigma_p^\# - \sigma)[u]^p_{B^\sigma_{p,p}} \leq C[u]^p_{B^{\sigma_p^\#}_{p,\infty}}. \qquad (15)$$

Suppose that (M, d, μ) *satisfies property* $(\widetilde{NE})$ *with* $\sigma_p > 0$, *then there exists a positive constant* C *such that for all* $u \in B^{\sigma_p}_{p,\infty}$,

$$C^{-1}\|u\|^p_{KS^{\sigma_p}_{p,\infty}} \leq \liminf_{\sigma \uparrow \sigma_p^\#}(\sigma_p - \sigma)[u]^p_{B^\sigma_{p,p}} \leq \limsup_{\sigma \uparrow \sigma_p}(\sigma_p - \sigma)[u]^p_{B^\sigma_{p,p}} \leq C\|u\|^p_{KS^{\sigma_p}_{p,\infty}}. \qquad (16)$$

Main Result

Theorem 5

If a metric measure space (M, d, μ) *admits a heat kernel* $\{p_t\}_{t>0}$ *satisfying* (5)*, then for any* $\sigma > 0$*,* $\mathcal{D}(E^{\sigma}_{p,\infty}) = B^{\sigma}_{p,\infty}$ *and for all* $u \in B^{\sigma}_{p,\infty}$*,*

$$E^{\sigma}_{p,\infty}(u) \asymp [u]^{p}_{B^{\sigma}_{p,\infty}} \text{ and } \|u\|^{p}_{KS^{\sigma}_{p,\infty}} \asymp \limsup_{t\to 0} \Psi^{\sigma}_{u}(t). \tag{17}$$

In addition, $\mathcal{D}(E^{\sigma}_{p,p}) = B^{\sigma}_{p,p}$ *and for all* $u \in B^{\sigma}_{p,p}$*,*

$$E^{\sigma}_{p,p}(u) \asymp [u]^{p}_{B^{\sigma}_{p,p}}. \tag{18}$$

Main Result

Theorem 6

If a metric measure space (M, d, μ) admits a heat kernel $\{p_t\}_{t>0}$ satisfying (5), then we have the following equivalences:

(i) $(\widetilde{KE}) \Longleftrightarrow (\widetilde{NE})$ *with the same* $\sigma > 0$,

(ii) $(KE) \Longleftrightarrow (NE)$ *with the same* $\sigma > 0$.

Lemma

Lemma 7

Suppose that (M, d, μ) *admits a heat kernel* $\{p_t\}_{t>0}$ *satisfying* (5). *Under the property* $(\widetilde{KE})$ *with* $\sigma > 0$, *we have*

$$\liminf_{t \to 0} \Psi_u^\sigma(t) \leq C \liminf_{r \to 0} \Phi_u^\sigma(r), \tag{19}$$

for all $u \in \mathcal{D}(E_{p,\infty}^\sigma)$.

Proof

Temporally fix $u \in \mathcal{D}(E^{\sigma}_{p,\infty})$. Without loss of generality, assume that $\liminf_{t\to 0} \Psi^{\sigma}_{u}(t) > 0$. For $t \in (0, R_0^{\beta^*})$,

$$\Psi^{\sigma}_{u}(t) = \Psi_1(t) + \Psi_2(t),$$

where

$$\Psi_1(t) = \frac{1}{t^{p\sigma/\beta^*}} \iint_{\{d(x,y)\leq \delta t^{1/\beta^*}\}} |u(x) - u(y)|^p p_t(x,y) d\mu(y) d\mu(x),$$

$$\Psi_2(t) = \frac{1}{t^{p\sigma/\beta^*}} \iint_{\{d(x,y)> \delta t^{1/\beta^*}\}} |u(x) - u(y)|^p p_t(x,y) d\mu(y) d\mu(x).$$

For $\Psi_1(t)$, when $d(x,y) \leq \delta t^{1/\beta^*}$, we have

$$\Psi_1(t) \leq \frac{c_3}{t^{p\sigma/\beta^*}} \iint_{\{d(x,y)\leq \delta t^{1/\beta^*}\}} \frac{|u(x) - u(y)|^p}{V(x, t^{1/\beta^*})} d\mu(y) d\mu(x)$$

$$\leq \frac{c_3 C_d \delta^{\alpha_1}}{t^{p\sigma/\beta^*}} \iint_{\{d(x,y)\leq \delta t^{1/\beta^*}\}} \frac{|u(x) - u(y)|^p}{V(x, \delta t^{1/\beta^*})} d\mu(y) d\mu(x) = C_1 \Phi^{\sigma}_{u}(\delta t^{1/\beta^*}),$$

where $C_1 = c_3 C_d \delta^{p\sigma + \alpha_1}$.

Proof

For $\Psi_2(t)$, by

$$p_t(x,y) \le C \exp\left(-c'\delta^{\frac{\beta^*}{\beta^*-1}}\right) p_{ct}(x,y).$$

we have

$$\Psi_2(t) \le A\Psi_u^\sigma(ct), \tag{20}$$

where

$$A = Cc^{p\sigma/\beta^*} \exp\left(-c'\delta^{\frac{\beta^*}{\beta^*-1}}\right) < \frac{1}{2}. \tag{21}$$

Combining $\Psi_1(t)$ and $\Psi_2(t)$, we have

$$\Psi_u^\sigma(t) \le C_1\Phi_u^\sigma(\delta t^{1/\beta^*}) + A\Psi_u^\sigma(ct). \tag{22}$$

Let $t = c^{-(n+1)}$. It follows that

$$\Psi_u^\sigma(c^{-(n+1)}) \le C_1\Phi_u^\sigma(\delta c^{-(n+1)/\beta^*}) + A\Psi_u^\sigma(c^{-n}).$$

Proof

Since $\limsup_{r\to 0} \Phi_u^\sigma(r) < \infty$, there exists M_0 such that

$$\sup_{n\geq 1} \Phi_u^\sigma(\delta c^{-n/\beta^*}) \leq M_0 < \infty.$$

Picking up $M_1 = \max\{C_1 M_0, \Psi_u^\sigma(c^{-1})\}$, we have

$$\Psi_u^\sigma(c^{-2}) \leq (1+A)M_1.$$

By induction, we have

$$\Psi_u^\sigma(c^{-n}) \leq M_1 \left(\frac{1-A^n}{1-A}\right).$$

Letting $n \to \infty$, we obtain

$$\liminf_{t\to 0} \Psi_u^\sigma(t) < \infty.$$

Proof

By property $(\widetilde{KE})$, there exists $t_0 > 0$ such that for all $t \in (0, t_0)$,

$$\Psi_u^\sigma(t) \leq \sup_{0<t<t_0} \Psi_u^\sigma(t) \leq 2 \limsup_{t\to 0} \Psi_u^\sigma(t) \leq C_2 \liminf_{t\to 0} \Psi_u^\sigma(t),$$

where C_2 comes from property $(\widetilde{KE})$. Therefore, we have from (22) that, for all small enough $t \in (0, \frac{t_0}{c})$,

$$\Psi_u^\sigma(t) \leq C_1 \Phi_u^\sigma(\delta t^{1/\beta^*}) + AC_2 \liminf_{t\to 0} \Psi_u^\sigma(t).$$

Fixing δ such that $AC_2 < 1$ with aforementioned requirement $A < \frac{1}{2}$, it follows that

$$\liminf_{t\to 0} \Psi_u^\sigma(t) \leq C_1 \liminf_{t\to 0} \Phi_u^\sigma(t) + AC_2 \liminf_{t\to 0} \Psi_u^\sigma(t),$$

which implies

$$\liminf_{t\to 0} \Psi_u^\sigma(t) \leq C_3 \liminf_{t\to 0} \Phi_u^\sigma(t),$$

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Fractal glue-ups

Next, we introduce the construction of *fractal glue-ups*. The definition is adapted from the idea of fractafold in [ST12] but not the same. Given a compact set $K \subset \mathbb{R}^d$ and a set F of similitudes on $\mathbb{R}^d$, we can define

$$K^F := \bigcup_{f \in F} f(K) \subset \mathbb{R}^d.$$

For our study, we need the following additional requirements.

1. Isometries: all similitudes in F have contraction ratio 1, that is, every similitude in F is an isometry of K.
2. Just-touching property: for any $f \neq g \in F$,
$$f(K) \bigcap g(K) = f(V_0) \bigcap g(V_0).$$
3. Condition (H): There exists a constant $C_H \in (0,1)$ such that, if $|x-y| < C_H \rho^m$ with $x \in f_1(K_w)$ and $y \in f_2(K_{\tilde{w}})$ for two words w and $\tilde{w}$ with the same length m, then $f_1(K_w)$ intersects $f_2(K_{\tilde{w}})$.
4. Connectedness: K^F is connected.

Fractal blow-up

We illustrate the fractal blow-up of the Sierpiński gasket K (the highlighted area) see [Strichartz98] in Fig.1.

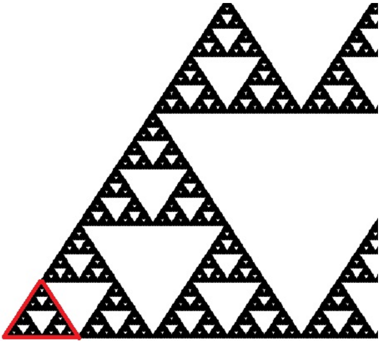

Figure: The blow-up of the Sierpiński gasket

Definition of discrete p-energy norms

Next, we consider discrete p-energy norms of K^F. Following the definitions for K in [GYZ22], define

$$E_n^{(p)}(u) := \sum_{x,y\in V_w,|w|=n} |u(x)-u(y)|^p \quad \text{and} \quad \mathcal{E}_n^{\sigma}(u) := \rho^{-n(p\sigma-\alpha)} E_n^{(p)}(u),$$

then the local p-energy norm on K^F is given by

$$E_n^{(p),F}(u) := \sum_{f\in F} \sum_{x,y\in V_w,|w|=n} |u\circ f(x) - u\circ f(y)|^p = \sum_{f\in F} E_n^{(p)}(u\circ f), \tag{23}$$

and

$$\mathcal{E}_n^{\sigma,F}(u) = \rho^{-n(p\sigma-\alpha)} E_n^{(p),F}(u).$$

Weak-monotonicity properties(VE)

Definition 8

We say that a fractal glue-up K^F satisfies property (VE) with $\sigma > 0$, if there exists $C > 0$ such that for all $u \in L^p(K^F, \mu^F)$,

$$\sup_{n\geq 0} \mathcal{E}_n^{\sigma,F}(u) \leq C \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u).$$

We say that a fractal glue-up K^F satisfies property $(\widetilde{VE})$ with $\sigma > 0$, if there exists $C > 0$ such that for all $u \in L^p(K^F, \mu^F)$,

$$\limsup_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u) \leq C \liminf_{n\to\infty} \mathcal{E}_n^{\sigma,F}(u).$$

Main Result

Theorem 9

Let K^F be a fractal glue-up, where K is a connected homogeneous p.c.f. self-similar set, then we have the following equivalences:

(i) $(\widetilde{VE}) \Longleftrightarrow (\widetilde{NE})$ *with the same* $\sigma > \alpha/p$ *and* $R_0 = 1$,

(ii) $(VE) \Longleftrightarrow (NE)$ *with the same* $\sigma > \alpha/p$ *and* $R_0 = 1$.

Main Result

Theorem 10

Let K^F be a fractal glue-up, where K is a connected homogeneous p.c.f. self-similar set satisfying property (E). Then K^F satisfies (NE) with $R_0 = 1$ and $B_{p,\infty}^{\sigma_p^{\#}(K^F)}(K^F) := B_{p,\infty}^{\sigma_p^{\#}}(K^F)$ contains non-constant functions. Also, the BBM convergence (15) holds for K^F with $R_0 = 1$, that is, for $R_0 = 1$, there exists a positive constant C such that for all $u \in B_{p,\infty}^{\sigma_p^{\#}}(K^F)$,

$$\begin{aligned} C^{-1}[u]^p_{B_{p,\infty}^{\sigma_p^{\#}}(K^F)} &\leq \liminf_{\sigma \uparrow \sigma_p^{\#}(K^F)} \left(\sigma_p^{\#}(K^F) - \sigma\right) [u]^p_{B^{\sigma}_{p,p}(K^F)} \\ &\leq \limsup_{\sigma \uparrow \sigma_p^{\#}(K^F)} \left(\sigma_p^{\#}(K^F) - \sigma\right) [u]^p_{B^{\sigma}_{p,p}(K^F)} \leq C[u]^p_{B_{p,\infty}^{\sigma_p^{\#}}(K^F)}. \end{aligned}$$

Furthermore, if K is a nested fractal and $K^F = K_\infty$, then (NE) and (KE) are satisfied with $R_0 = \infty$, and the Gagliardo-Nirenberg inequality (24) holds true.

Lemma

Next, we introduce the following embedding inequality given by Baudoin, using our notions.

Lemma 11 (Baudoin22)

Suppose that property (NE) holds with $R_0 = \infty$ *for the metric measure space* (M, d, μ)*, and there exists* $R > 0$ *such that* $\inf_{x\in M} \mu(B(x,R)) > 0$*. When* $p\sigma_p^\# \neq \alpha_1$*, where* α_1 *is from (3), let* $q = \frac{p\alpha_1}{\alpha_1 - p\sigma_p^\#}$*. For* $r, s \in (0, \infty]$*,* $\theta \in (0,1]$ *satisfying*

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s},$$

there exists a constant $C > 0$ *such that for any* $f \in B_{p,\infty}^{\sigma_p^\#}$*,*

$$\|f\|_r \leq C(\|f\|_p + [f]_{B_{p,\infty}^{\sigma_p^\#}})^\theta \|f\|_s^{1-\theta}. \tag{24}$$

Thank you!