Boundary value problems for harmonic functions on domains in p.c.f. self-similar sets

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Dirichlet problem and Poisson kernel

Let B be a unit ball in $\mathbb{R}^n$, let $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ be the Laplace operator.

For a continuous function f on the boundary ∂B, the **Dirichlet problem**

$$\begin{cases} \Delta u = 0, & \text{in } B \\ u = f, & \text{on } \partial B \end{cases}$$

has a unique solution u given by

$$u(x) = \int_{\partial B} f(\zeta) P(x, \zeta) d\sigma(\zeta), \qquad x \in B,$$

where σ is the normalized surface measure on ∂B and $P(x, \zeta) = \frac{1-|x|^2}{|x-\zeta|^n}$ is called the **Poisson kernel**.

The probabilistic point of view

Let $\{X_t\}_{t\geq 0}$ be the standard Brownian motion starting at some point $x \in B$.

Denote by $\tau := \inf\{t > 0 : X_t \in \partial B\}$ the exit time of X_t from B.

The **hitting probability** is

$$\mathbb{P}_x(X_\tau \in A) = \int_A P(x,\zeta)d\sigma(\zeta),$$

for any Borel set $A \subseteq \partial B$.

The measure $P(x,\zeta)d\sigma(\zeta)$ is also called the harmonic measure of the unit ball B.

Dirichlet forms on fractals

Let (M, d, μ) be a metric measure space.

A **Dirichlet form** is a closed densely-defined nonnegative symmetric bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, d\mu)$ satisfying the Markovian property, i.e. $u \in \mathcal{F}$ implies $\bar{u} = (u \vee 0) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\bar{u}) \leq \mathcal{E}(u)$.

Example: $\mathcal{E}(u) = \int_\Omega |\nabla u|^2 dx$ on an Euclidean domain Ω.

Dirichlet form has an infinitesimal generator $\mathcal{L}$ satisfying $\mathcal{E}(u, v) = (-\mathcal{L}u, v)$, called the Laplacian.

Many examples are constructed on **fractals**: post-critically finite (p.c.f.) self-similar sets (including nested fractals), generalized Sierpinski carpets.

p.c.f. self-similar sets

Let $N \geq 2$, $\{F_i\}_{i=1}^N$ be a collection of contractions on (X, d). The self-similar set associated with the iterated function system (IFS) $\{F_i\}_{i=1}^N$:

$$K = \bigcup_{i=1}^{N} F_i(K).$$

Let $\Sigma = \{1, \ldots, N\}$ be the alphabets. Let $\pi : \Sigma^\infty \to K$ be defined by $\{x\} = \{\pi(\omega)\} = \bigcap_{n\geq 1} F_{[\omega]_n}(K)$ with $[\omega]_n = \omega_1 \cdots \omega_n$. Following Kigami, the critical set $\mathcal{C}$ and post-critical set $\mathcal{P}$

$$\mathcal{C} = \pi^{-1}\left(\bigcup_{1\leq i<j\leq N} (F_i(K) \cap F_j(K)) \right), \qquad \mathcal{P} = \bigcup_{m\geq 1} \sigma^m(\mathcal{C}),$$

where $\sigma : \Sigma^\infty \to \Sigma^\infty$ is the left shift operator.

p.c.f. self-similar sets

If $\mathcal{P}$ is finite, we call $\{F_i\}_{i=1}^N$ a post-critically finite (p.c.f.) IFS, and K a p.c.f. self-similar set. The boundary of K is defined by $V_0 = \pi(\mathcal{P})$. We also inductively denote

$$V_n = \bigcup_{i\in\Sigma} F_i(V_{n-1}), \qquad V_* = \bigcup_{n=0}^{\infty} V_n.$$

It is clear that $\{V_n\}_{n\geq 0}$ is an increasing sequence of sets and K is the closure of V_*. We always assume that (K, d) is connected.

p.c.f. self-similar sets

Our basic assumption on a p.c.f. self-similar set K is the existence of a regular self-similar resistance form $(\mathcal{E}, \mathcal{F})$ with domain $\mathcal{F} = \{u \in C(K) : \ \mathcal{E}[u] := \mathcal{E}(u,u) < \infty\}$:

$$\mathcal{E}[u] = \sum_{i=1}^{N} \frac{1}{r_i} \mathcal{E}[u \circ F_i], \qquad u \in \mathcal{F}, \tag{1}$$

where $0 < r_i < 1, i = 1, \ldots, N$ are called energy renormalizing factors. By iterating (1), we see that for any $n \geq 1$,

$$\mathcal{E}[u] = \sum_{|\omega|=n} \frac{1}{r_\omega} \mathcal{E}[u \circ F_\omega], \qquad u \in \mathcal{F}, \tag{2}$$

where $r_\omega = r_{\omega_1} \cdots r_{\omega_n}$ for $\omega = \omega_1 \cdots \omega_n$. We call $\mathcal{E}_{F_\omega(K)}[u] := \frac{1}{r_\omega} \mathcal{E}[u \circ F_\omega]$ the energy of u on the cell $F_\omega(K)$.

the nested fractals

The nested fractals, introduced in [Lindstrøm 1990], is a class of p.c.f. fractals generated by an iterated function system (IFS) $\{F_i\}_{i=1}^N$ on $\mathbb{R}^d$ with a common contraction ratio and is

- **connected**
- **symmetric**
- **OSC**
- **nesting**

On a nested fractal, there exists a unique symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ (existence by Lindstrøm [Lindstrøm 1990] and uniqueness by Sabot [Sabot 1997]) satisfying the energy self-similar identity: for any $u \in \mathcal{F}$, $u \circ F_i \in \mathcal{F}$ for $1 \leq i \leq N$ and

$$\mathcal{E}(u) = \frac{1}{r} \sum_{i=1}^{N} \mathcal{E}(u \circ F_i).$$

examples of nested fractals

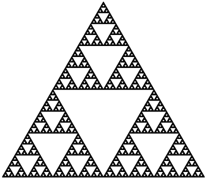


Figure: Sierpinski gasket ($r = \frac{3}{5}$)

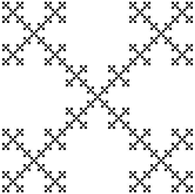


Figure: Vicsek set ($r = \frac{1}{3}$)

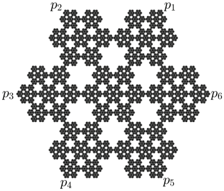


Figure: Lindstrøm snowflake($r \approx 0.543$)

resistance forms on p.c.f. fractals

We say a function $h \in \mathcal{F}$ harmonic in K if

$$\mathcal{E}[h] = \inf\{\mathcal{E}[u] : \ u \in \mathcal{F}, u|_{V_0} = h|_{V_0}\}.$$

Let A, B be two disjoint nonempty closed subsets of K, the effective resistance $R(A, B)$ between A and B is defined as

$$R(A, B)^{-1} := \inf\{\mathcal{E}[u] : \ u \in \mathcal{F}, u|_A = 0, u|_B = 1\}.$$

The infimum is attained by a unique function which is harmonic in $K \setminus (A \cup B)$. When we only consider points, by setting $R(x, x) = 0$ for all $x \in K$, the resistance $R(\cdot, \cdot)$ is a metric on K, which is called the effective resistance metric. It is known that

$$\mathrm{diam}_R(F_\omega(K)) \asymp r_\omega, \qquad \text{for any finite word } \omega.$$

some notations in electric networks

Let G be a finite set, and let $g : G \times G \to \mathbb{R}$ be a nonnegative function such that

$$g(p,q) = g(q,p),\ \ g(p,p) = 0, \quad p,q \in G.$$

For $p,q \in G$, we write $p \sim q$ if $g(p,q) > 0$. We always assume that (G,g) is connected, and call (G,g) an electric network. For $u \in \ell(G)$, we define the energy of u on (G,g) to be

$$\mathcal{E}_G[u] := \frac{1}{2} \sum_{p,q \in G} g(p,q)(u(p) - u(q))^2.$$

Then $(\mathcal{E}_G, \ell(G))$ is a resistance form on G.

some notations in electric networks

For $u \in \ell(G)$, we define the Neumann derivative of u (flux of ∇u, the flow associated with u) at some vertex $p \in G$ as

$$(du)_p = \sum_{q \in G} g(p,q)(u(p) - u(q)). \tag{3}$$

Then clearly, for $u, v \in \ell(G)$,

$$\sum_{p \in G} v(p)(du)_p = \sum_{p \in G} u(p)(dv)_p, \tag{4}$$

and in particular,

$$\sum_{p \in G} (du)_p = 0. \tag{5}$$

some notations in electric networks

For a resistance form $(\mathcal{E}, \mathcal{F})$ on a self-similar set K, it is known that the trace of $\mathcal{E}[\cdot]$ to a nonempty finite set $V \subset K$ is an electric network (V, g) determined by

$$\sum_{p,q \in V} g(p,q)(u(p) - u(q))^2 = \min\{\mathcal{E}[v] : v \in \mathcal{F}, v|_V = u\}, \quad u \in \ell(V),$$

while the unique function v minimizing the right hand side is harmonic in $K \setminus V$.

boundary value problem in domains of p.c.f. sets

For a given p.c.f. self-similar set K equipped with a local regular self-similar Dirichlet form, we are concerned with the boundary value problems for harmonic functions on a domain Ω in K (which means Ω is a nonempty open connected subset of K).

$$\begin{cases} \mathcal{E}(u,v) = 0, & \forall v \in \mathcal{F}_0(\Omega) \\ u = f, & \text{on } \partial\Omega \end{cases}$$

We mainly focus on **two problems**:

- find the exact description of the hitting probability from a point in Ω to the boundary;
- estimate the energy of a harmonic function generated by its boundary values.

Known results

[Owen-Strichartz 2012] initiated the study of boundary value problems for domains of the standard Sierpinski Gasket (SG).

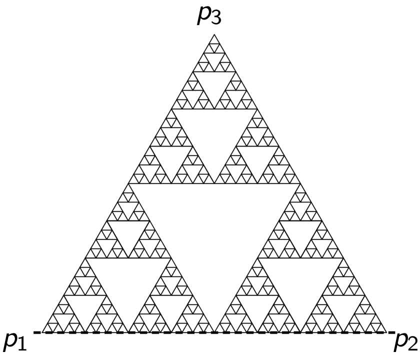


Figure: A domain in SG with a bottom line boundary

Known results

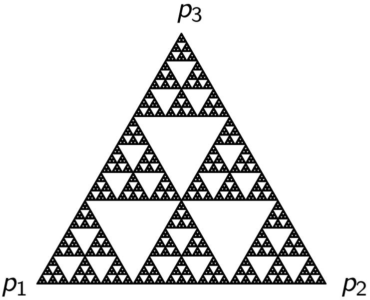
It is shown in [Owen-Strichartz 2012] that the harmonic measure with respect to p_3 for the above domain is the uniform (Lebesgue) measure on the bottom line $L = \overline{p_1 p_2}$, i.e.

$$u(p_3) = \int_L f(x)dx.$$

The key technique they employ is decomposition of a function f on the boundary by using **Haar basis**.
Guo, Kogan, Qiu and Strichartz [Guo et al. 2014] extended this result to more general domains with an arbitrary horizontal cut in SG.

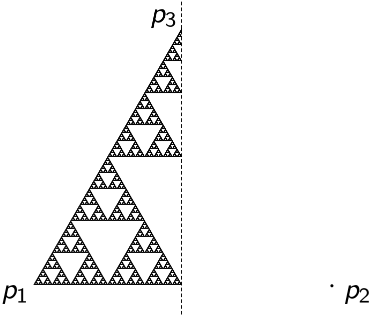
Known results

For the level-3 SG, Cao and Qiu [Cao-Qiu 2020] had a very detailed investigation on the domains with horizontal cut. In particular, for the domain with bottom line $L = \overline{p_1 p_2}$ as boundary, they showed that the harmonic measure with respect to p_3 is a self-similar measure on L with weight $\mu = \left(\frac{6+\eta}{18+4\eta}, \frac{6+2\eta}{18+4\eta}, \frac{6+\eta}{18+4\eta}\right)$ with $\eta = \frac{\sqrt{2353}-15}{14} \approx 2.3934$.



Known results on SGs

[Li-Strichartz 2014] studied the half domain of SG.

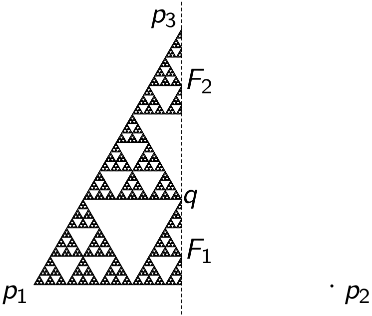


They obtained the harmonic measure with respect to p_1 is

$$\mu = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} \delta_{F_{3^n 1}(p_2)}.$$

Known results on SGs

[Cao-Qiu 2020] also considered the level-3 SG.



They obtained the harmonic measure with respect to p_1 is

$$\mu = \frac{2}{7}\delta_q + \sum_{|\omega|=1}^{\infty} \mu_\omega \delta_{F_\omega(q)}, \qquad \mu_1 = \frac{4}{7}, \mu_2 = \frac{1}{7}.$$

domains in p.c.f. fractals

Let $(K, \{F_i\}_{i=1}^N)$ be a p.c.f. fractal. For $P \geq 1$, let $\{\Omega_1, \Omega_2, \ldots, \Omega_P\}$ be a vector of connected open subsets of K with nonempty boundary $D_i := \partial\Omega_i$. We assume $\{(\Omega_i, D_i)\}_{1 \leq i \leq P}$ satisfy the following **boundary graph-directed condition (BGD)**:

for $1 \leq i \leq P$ and $1 \leq k \leq N$, if $\Omega_i \cap F_k(K) \neq \emptyset$ and $D_i \cap F_k(K) \neq \emptyset$, then there exists $1 \leq j \leq P$ such that

$$\Omega_i \cap F_k(K) = F_k(\Omega_j), \qquad D_i \cap F_k(K) = F_k(D_j).$$

domains satisfying BGD

Example 1.

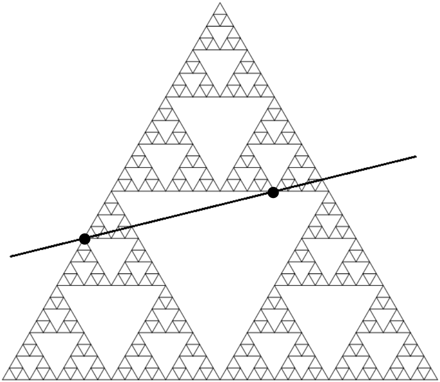


Figure: domains in the Sierpinski gasket

domains satisfying BGD

Example 2.

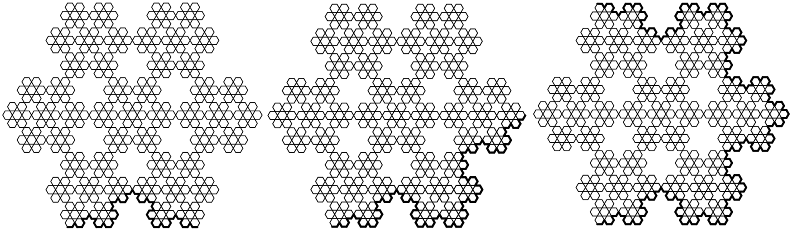


Figure: domains with graph-directed boundary in the Lindstrøm's snowflake

domains satisfying BGD

Example 3.

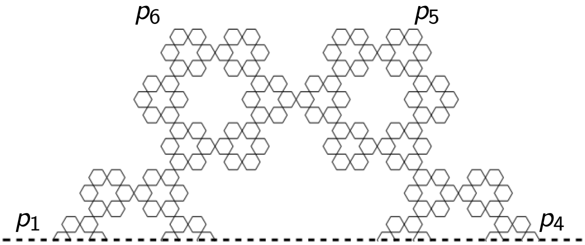


Figure: a half domain in the hexagasket

domains satisfying BGD

Example 4.

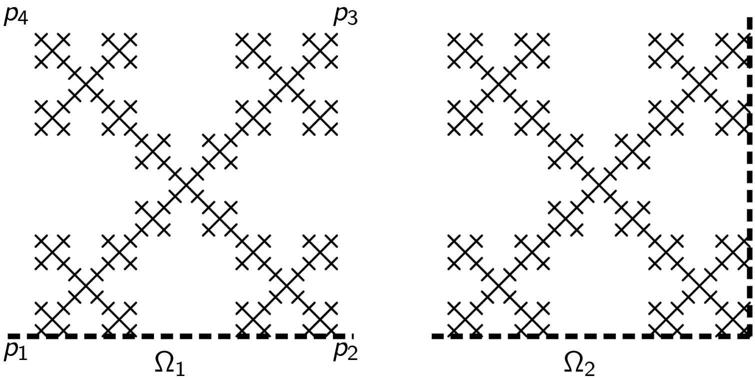


Figure: A couple of domains in the Vicsek set

notations for graph-directed self-similar sets

Let $(\mathcal{A}, \Gamma)$ be a directed graph with $\mathcal{A} = \{1, \dots, P\}$ and edges Γ. For $\gamma \in \Gamma$, if γ is a directed edge from i to j for some $i, j \in \mathcal{A}$, we denote by $I(\gamma) = i$ and $T(\gamma) = j$ the **initial vertex** and the **terminal vertex** separately.

For $i, j \in \mathcal{A}$, denote $\Gamma_i = \{\gamma \in \Gamma : \ I(\gamma) = i\}$ and $\Gamma_{i,j} = \{\gamma \in \Gamma : \ I(\gamma) = i, T(\gamma) = j\}$. Then each edge γ is associated with a contractive map Φ_γ and

$$D_i = \bigcup_{j=1}^{P} \bigcup_{\gamma \in \Gamma_{i,j}} \Phi_\gamma(D_j), \qquad 1 \leq i \leq P.$$

Let $m \geq 1$, a finite word $\gamma = \gamma_1 \gamma_2 \cdots \gamma_m$ with $\gamma_i \in \Gamma$ for $i = 1, \dots, m$ is called admissible if $T(\gamma_i) = I(\gamma_{i+1})$ for any $i = 1, \dots, m-1$; write $I(\gamma) = I(\gamma_1)$, $T(\gamma) = T(\gamma_m)$.

notations for graph-directed self-similar sets

We will also use the notation of infinite admissible words $\gamma = \gamma_1\gamma_2\cdots$ with $T(\gamma_i) = I(\gamma_{i+1})$ for all $i \geq 1$. We denote by Γ_∞ the collection of all infinite admissible words and $\Gamma_\infty(i) = \{\gamma = \gamma_1\gamma_2\cdots \in \Gamma_\infty : \ I(\gamma_1) = i\}$ for $i = 1, \ldots, P$.
For $\gamma = \gamma_1\gamma_2\cdots, \eta = \eta_1\eta_2\cdots \in \Gamma_\infty$ with $\gamma \neq \eta$, let $\gamma \wedge \eta$ be the common prefix of γ and η, then

$$|\gamma \wedge \eta| = \min \{i \geq 1 : \ \gamma_i \neq \eta_i\} - 1.$$

Define

$$\rho(\gamma, \eta) = \begin{cases} 2^{-|\gamma\wedge\eta|}, & \gamma \neq \eta, \\ 0, & \gamma = \eta. \end{cases}$$

Then ρ is a metric on Γ_∞ and (Γ_∞, ρ) is a complete metric space.

properties of the BGD domains

Proposition (G.-Qiu 2024)

Assume $\{\Omega_i\}_{i=1}^P$ *satisfies the BGD condition.*
(i). If $\Omega_i \cap V_0 \neq \emptyset$*, then* $\Omega_j \cap V_0 \neq \emptyset$ *provided that* $\Gamma(i,j) \neq \emptyset$*;*
(ii). There exists $n_0 \geq 1$ *such that* $\Omega_{T(\gamma)} \cap V_0 \neq \emptyset$ *for all* $n \geq n_0$ *and* $\gamma \in \Gamma_n$*.*

Proposition (G.-Qiu 2024)

Assume $\{\Omega_i\}_{i=1}^P$ *satisfies the BGD condition. Then each* Ω_i *is arcwise connected.*

geometric boundary and resistance boundary

Let Ω be a domain in K. The geometric boundary of Ω in K is defined through the original metric in K.
For a function $u \in C(\Omega)$, denote the energy of u on Ω as $\mathcal{E}_\Omega[u]$.
Denote $\mathcal{F}_\Omega = \{u \in C(\Omega) : \ \mathcal{E}_\Omega[u] < \infty\}$. It is direct to check that $(\mathcal{E}_\Omega, \mathcal{F}_\Omega)$ is a resistance form on Ω. Define the effective resistance metric $R_\Omega(x, y)$ for two points x, y in Ω with respect to $\mathcal{E}_\Omega$: for $x, y \in \Omega$ and $x \neq y$,

$$R_\Omega(x, y)^{-1} := \inf\{\mathcal{E}_\Omega[u] : \ u \in \mathcal{F}_\Omega, u(x) = 0, u(y) = 1\}.$$

Then $R_\Omega(\cdot, \cdot)$ is a metric on Ω. Let $\widetilde{\Omega}$ be the completion of Ω under R_Ω, and denote $\partial\Omega = \widetilde{\Omega} \setminus \Omega$, the resistance boundary of Ω.
Recall that there is another resistance metric $R(\cdot, \cdot)$ on Ω inherited from that on K.

properties of the BGD domains

Proposition (G.-Qiu 2023)

Let $A \subset \Omega$ be a nonempty closed subset. Then there exists $C > 1$ depending on A such that

$$R(x,y) \le R_\Omega(x,y) \le CR(x,y), \quad \forall x,y \in A.$$

In addition, (A, R_Ω) is homeomorphic to (A, R) and (A, d).

Proposition (G.-Qiu 2024)

There exists $n_1 \ge 1$ such that for each Ω_i with $\Omega_i \cap V_1 \ne \emptyset$ and $x, y \in \Omega_i \cap V_1$, there exists a chain of n_1-cells $\{F_{\omega^{(k)}}(K)\}_{k=1}^m$ in Ω_i connecting x and y.

properties of the BGD domains

Theorem (G.-Qiu 2024)

Each (Ω_i, R_{Ω_i}) is a bounded metric space.

Theorem (G.-Qiu 2024)

For $i = 1, \ldots, P$, $(\partial\Omega_i, R_{\Omega_i})$ is homeomorphic to $(\Gamma_\infty(i), \rho)$.

The flux transfer matrices

Let $(\mathcal{A}, \Gamma)$ be the directed graph induced from the BGD condition. For each $\gamma \in \Gamma(i,j)$, there is a contraction map Φ_γ such that $\Phi_\gamma(\Omega_j) \subset \Omega_i$. In the following, we associate each γ with a $Q \times Q$ real matrix M_γ, whose (k,ℓ)-entry represents:

the flux of the unit flow on $\widetilde{\Omega}_i$ from $\partial\Omega_i$ to p_k through $\Phi_\gamma(p_\ell)$ outwards from $\Phi_\gamma(\Omega_j)$.

For any $1 \leq k \leq Q$, if $p_k \notin \Omega_i$, we simply set the k-th row of M_γ to be zeros; otherwise, if $p_k \in \Omega_i$, let φ be the realization of $R_{\Omega_i}(\partial\Omega_i, p_k)$. Let

$$v_k := R_{\Omega_i}(\partial\Omega_i, p_k)\varphi,$$

then v_k satisfies $(dv_k)_{p_k} = 1$. Denote the restriction of the function v_k on $\Phi_\gamma(\Omega_j)$ as $\tilde{v}_k$ and define $M_\gamma(k,\ell) = (d\tilde{v}_k)_{\Phi_\gamma(p_\ell)}$.

The hitting probabilities

Lemma (G.-Qiu 2024)

For $1 \leq i \leq P$ *and* $1 \leq k \leq Q$ *such that* $p_k \in \Omega_i \cap V_0$, *we have*

$$\sum_{\ell=1}^{Q} M_\gamma(k,\ell) > 0, \quad \forall \gamma \in \Gamma(i),$$

and

$$\sum_{\gamma \in \Gamma(i)} \sum_{\ell=1}^{Q} M_\gamma(k,\ell) = 1.$$

The hitting probabilities

Definition

For $\gamma = \gamma_1 \cdots \gamma_m \in \Gamma_m(i)$, write $M_\gamma = M_{\gamma_1} \cdots M_{\gamma_m}$. We define

$$\mu_{i,k}(\partial\Omega_\gamma) = \mathbf{e}_k^t M_\gamma \mathbf{1}.$$

Note that $\mu_{i,k}(\partial\Omega_\gamma)$ is the summation of the k-th row of M_γ.

Proposition

For $p_k \in \Omega_i \cap V_0$, $\mu_{i,k}$ extends to be a probability measure on $\partial\Omega_i$. Moreover, we have the identity

$$\mu_{i,k} = \sum_{\gamma \in \Gamma(i), 1 \leq \ell \leq Q} M_\gamma(k,\ell)\mu_{T(\gamma),\ell} \circ \theta_\gamma^{-1}.$$

first main result

Theorem (G.-Qiu 2024)

For $p_k \in \Omega_i \cap V_0$, the probability measure $\mu_{i,k}$ in Definition 8 is the hitting probability of p_k to the R-boundary $\partial\Omega_i$. Consequently, for any $f \in C(\partial\Omega_i)$, the unique harmonic function u on Ω_i generated by f, i.e. $u|_{\partial\Omega_i} = f$, satisfies

$$u(p_k) = \int_{\partial\Omega_i} f(x) d\mu_{i,k}(x).$$

property of the hitting measures

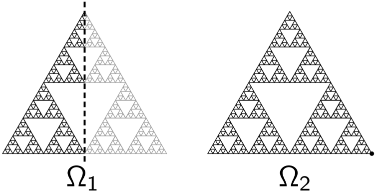
Theorem

For each $i \in \mathcal{A}$, *assume* $p, p' \in \Omega_i \cap V_0$ *and let* $\mu_{i,p}, \mu_{i,p'}$ *be the associated probability measures. Then there exists a constant* $C > 0$ *such that for any measurable set* $E \subset \partial\Omega_i$,

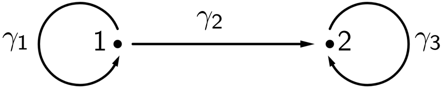
$$C^{-1}\mu_{i,p}(E) \leq \mu_{i,p'}(E) \leq C\mu_{i,p}(E).$$

Examples

Example 1:

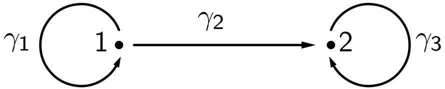


$\mathcal{A} = \{1, 2\}$, $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$.



Examples

$\mathcal{A} = \{1, 2\}$, $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$.



One may compute

$$M_{\gamma_1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; M_{\gamma_2} = \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_{\gamma_3} = \begin{pmatrix} 2/3 & 0 & 1/3 \\ 0 & 0 & 0 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

The hitting probability from p_1 to the boundary $\partial\Omega$ is

$$\mu = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} \delta_{F_{3^n 1}(p_2)}.$$

Examples

Example 2:

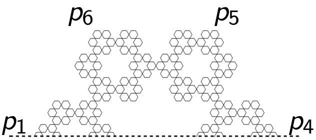


Figure: a half domain in the hexagasket

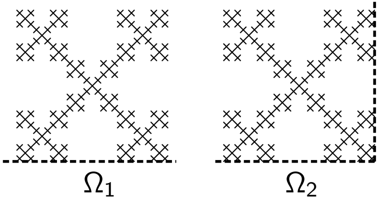
The associated flux transfer matrices are

$$M_{\gamma_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 0 \end{pmatrix}, M_{\gamma_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 1/3 \end{pmatrix}.$$

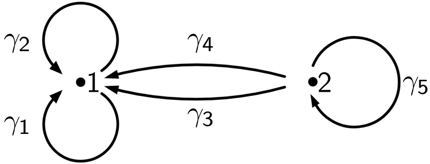
The hitting probability from p_5 (or p_6) is a twisted $(1/3, 2/3)$-self-similar measures on $\partial\Omega$.

Examples

Example 2:



Directed graph $\mathcal{A} = \{1, 2\}$ and $\Gamma = \{\gamma_i\}_{i=1}^{5}$.



Examples

The associated flux transfer matrices are

$$M_{\gamma_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 \end{pmatrix}, M_{\gamma_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}, M_{\gamma_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{13+t}{26+14t} & 0 \end{pmatrix},$$

$$M_{\gamma_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{13+t}{26+14t} \end{pmatrix}, M_{\gamma_5} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6t}{13+7t} \end{pmatrix}, \quad \text{where } t = \frac{\sqrt{69}-2}{5}.$$

For Ω_1, the hitting probability from p_3 (or p_4) is the $(1/2, 1/2)$-self-similar measures on $\partial\Omega_1$.
For Ω_2, the hitting probability μ from p_4 to $\partial\Omega_2$ is: for any $k \geq 0$, μ restricted on the boundary of $F_{2^k 1}(\Omega_1)$ is $(1/2, 1/2)$-self-similar measure with total weight $\left(\frac{6t}{13+7t}\right)^k \left(\frac{13+t}{26+14t}\right)$.

The energy estimates

For given function f on the boundary $\partial\Omega$, let u be its harmonic extension in Ω.

Want: express the energy of u via its boundary value f.

Classical case:

Let $B = \{x \in \mathbb{R}^2 : \ |x| < 1\}$ be the unit disc and $S = \partial B$. For $f \in L^1(S)$, let u be harmonic extension of f in B, then

$$\iint_B |\nabla u(x)|^2 dx = \frac{1}{16\pi} \int_S \int_S \frac{|f(\theta) - f(\vartheta)|^2}{\sin^2(\frac{\theta-\vartheta}{2})} d\theta d\vartheta.$$

The integral on the RHS is known as the **Douglas integral**.

The energy estimates on binary trees

Random walks on binary trees .

Theorem (Kigami 2010, Theorem 5.6)

Let T be a binary tree with energy form $(\mathcal{E}, \mathcal{F})$

$$\mathcal{E}_T[f] = \Sigma_{w \in T} \Sigma_{i=1,2} \frac{1}{r_{wi}} (f(v) - f(vi))^2,$$

and $\mathcal{F} = \{f \mid \mathcal{E}_T[f] < \infty\}$. Let $\Sigma = \partial T$ be its Martin boundary. For $w \in T$, let u_w be the average of u on Σ_w w.r.t. the hitting probability ν. Then the induced form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on Σ is

$$\mathcal{E}_\Sigma[u] = \Sigma_{w \in T} \frac{|u_{w1} - u_{w2}|^2}{r_{w1} + R_{w1} + r_{w2} + R_{w2}},$$

with $\mathcal{F}_\Sigma = \{u \mid \mathcal{E}_\Sigma[u] < \infty\}$.

The energy estimates on hyperbolic graphs

Random walks on (hyperbolic) augmented trees.

Theorem (Kong-Lau-Wong 2017, Theorem 1.4)

Let $\mathcal{E}_X$ be the energy form of the λ-natural random walk on the augmented tree (X, E) of an IFS satisfying open set condition. Then the induced form on the Martin boundary $\partial X (= K)$ satisfies

$$\mathcal{E}_K[u] \asymp \int_K \int_K \frac{|u(x) - u(y)|^2}{|x - y|^{\alpha+\beta}} d\nu(x) d\nu(y),$$

where the positive constant in "$\asymp$" is independent of u.

The energy estimates

Lemma

Let u be a harmonic function on K. We have

$$\mathcal{E}[u] \asymp \sum_{p,q \in V_0} |u(p) - u(q)|^2 \asymp \sum_{p \in V_0} |(du)_p|^2 ,$$

where the positive constants in the two "$\asymp$"'s are independent of u.

The energy estimates

For $f \in C(\partial\Omega)$ and $p \in V^{(\gamma)}$, we denote

$$f_{\gamma,p} = \int_{\partial\Omega_{T(\gamma)}} f \circ \theta_\gamma d\mu_{T(\gamma),p}.$$

Theorem (G.-Qiu 2024)

Assume $\Omega \cap V_0 \neq \emptyset$. For $f \in C(\partial\Omega)$, let u be the harmonic extension of f in Ω. Then

$$\mathcal{E}_\Omega[u] \asymp \sum_{m=0}^{\infty} \sum_{\gamma \in \Gamma_m} \frac{1}{r_\gamma} \sum_{\xi,\eta:\ \xi^- = \eta^- = \gamma} \sum_{p \in V^{(\xi)},\ q \in V^{(\eta)}} \left(f_{\xi,p} - f_{\eta,q}\right)^2,$$

where the constant in "$\asymp$" does not depend on u or f.

Thank You !!