# On gradient blowup profiles for nonlinear heat equations

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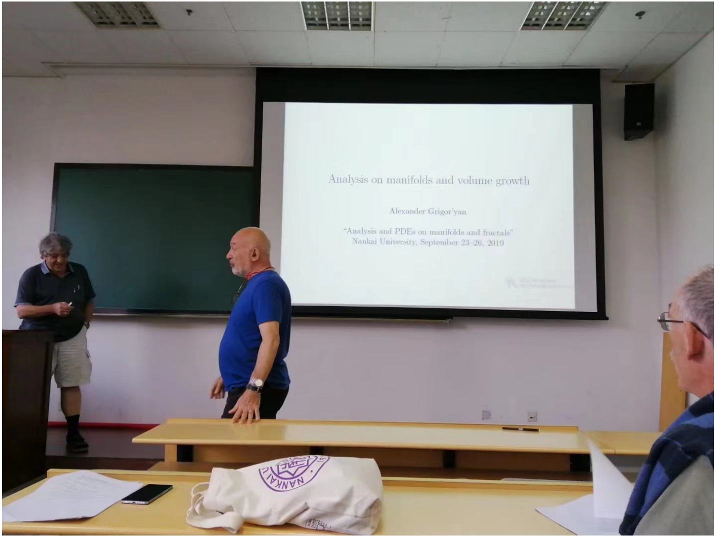
# Heat Kernel: a personal journey

This is a wordy page in a (Chinese) talk for the only non-Chinese-speaking audience*

- Almost all the participants of this workshop agree with the famous slogan "The Ubiquitous Heat Kernel" by Jay Jorgensen and Serge Lang. (Me too!)
- "Ubiquitous" is also justified by the fact that we come from all major cities in China: Hangzhou, Beijing, Guangzhou, Tianjin, Chongqing $\cdots$ (+ Bielefeld from Europe!)
- My first harmonic analysis training started with Prof. Jun CAO's presentations at our Alma Mater BNU about the Davies-Gaffney estimates (integrated heat semigroup bounds) in the adapted Hardy space paper of Hofmann-Mayboroda-McIntosh.
& Prof. Liguang LIU was the Teaching Assistant of our Mathematical Analysis class.
- On the table of my academic sister Li CHEN (supervised by Coulhon & Auscher), "Heat Kernel and Analysis on Manifolds" by Prof. Grigor'yan served as Red Bible.
- My first domestic conference was the one organised by Prof. Yuhua SUN in 2017.
& My first visit abroad was at Hokkaido and hosted by Aikawa and Masamune.
& My first friend in Nanjing is Prof. Xueping HUANG who influenced me very much.
- This personal journey shows that I benefited a lot from the community in this room. So I feel happy and lucky to have the opportunity to be here, right by West Lake.

*My spoken english is poor and I beg his pardon for the dichotomy: "read" slides or lake.


Analysis on manifolds and volume growth
Alexander Grigor'yan
"Analysis and PDEs on manifolds and fractals"
Nankai University, September 23–26, 2019

Let $L = -\mathrm{div} A\nabla$, with $A = A(x)$ being the (complex) elliptic coefficient matrix, $x \in \mathbb{R}^n$.

- Maximal Regularity of $\partial_t u + Lu = f$: for (certain) $1 < p < \infty$,

$$\|Lu\|_{L^2_t L^p_x((0,\infty)\times\mathbb{R}^n)} \lesssim \|f\|_{L^2_t L^p_x((0,\infty)\times\mathbb{R}^n)}.$$

  The mapping $f \mapsto Lu$ is a singular integral with operator-valued kernel.

- Conical Maximal Regularity (CMR) of $\partial_t u + Lu = f$: for (certain) $1 < p < \infty$,

$$\|Lu\|_{T^p_2((0,\infty)\times\mathbb{R}^n)} \lesssim \|f\|_{T^p_2((0,\infty)\times\mathbb{R}^n)}.$$

  Here $T^p_2$ is the so-called tent space of Coifman-Meyer-Stein introduced in 1983. CMR is actually the motivating ex. of my thesis Operator Theory on Tent Spaces (supervised by Auscher and defended in 2015), with the aim for developing functional and harmonic analytic tools for elliptic BVP and parabolic regularity.

- Both MR and CMR theories rely crucially on the heat "kernel" bounds. See e.g. Coulhon-Lamberton, Hieber-Prüss, Auscher-Kriegler-Monniaux-Portal.

**Is Heat Kernel also helpful for singularity of NLH, just for $L = -\Delta$ and $f(u) = u^p$?**

- Talk today (shall end with "Mode Dynamics via Mehler Kernel") is about the Singularity Formation of Parabolic Equations (via Heat Kernel Estimates).
- Alternative title for PDEs and Physical Modelling: Gradient profile for the reconnection of vortex lines with the boundary in type-II superconductors.

# Quenching Problem

- Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $N \geq 1$. Let $\beta > 0$. In this talk, we are interested in the quenching problem modelled on the nonlinear heat equation on $\Omega$

$$\frac{\partial h}{\partial t} = \Delta h - F(h), \tag{1}$$

where

$$F = F_\beta = \frac{1}{h^\beta} + \widetilde{F} \in C^\infty(\mathbb{R}_+), \quad \text{with } \mathbb{R}_+ = (0, \infty).$$

We shall assume that $\widetilde{F}$ satisfies

$$\widetilde{F}(h) = o\left(\frac{1}{h^\beta}\right) \quad \text{and} \quad \widetilde{F}'(h) = o\left(\frac{1}{h^{\beta+1}}\right), \quad \text{as } h \to 0, \tag{2}$$

and $h$ is subject to initial data $h_0 = h(\cdot, 0) > 0$ and the Dirichlet BC $h \equiv 1$ on $\partial\Omega$.

- Finite time *quenching* or *extinction* of a solution $h$ to the Cauchy problem for (1) at $x_0 \in \Omega$, means that for some $T \in \mathbb{R}_+$, $h$ has limit value 0 at $(x_0, T)$. We remark that the perturbative assumption (2) is typically satisfied by

$$\widetilde{F} \equiv 0, \quad \widetilde{F}(h) = e^{-h}, \quad \textit{or} \quad \widetilde{F}(h) = \frac{1}{h^{\beta'}} \text{ for some } \beta' < \beta.$$

In the quenching scenario $h \to 0$, apart from $F \in C^\infty(\mathbb{R}_+)$ there is no need (and no gain) to prescribe the behaviour of $F(h)$ for $h \geq 1$. Equation (1) can also be considered on the whole space $\mathbb{R}^N$, assuming decay of $F$ and $F'$ and growth of $h$.

# Blowup Problem

- By introducing the following family of transformations

$$u(x,t) = u(\alpha, x, t) = \frac{\alpha^{\frac{\alpha}{\beta+1}}}{h(x,t)^{\alpha}} \quad (\alpha > 0) \tag{3}$$

for (1) we are then led to study the blowup problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u - a\frac{|\nabla u|^2}{u} + f(u), \\ f(u) &= \alpha^{\frac{\beta}{\beta+1}} u^{1+\frac{1}{\alpha}} F(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) = u^p + \widetilde{f}(u), \end{aligned} \tag{4}$$

where $u > 0$, $u \equiv 1$ on $\partial\Omega$, $(a, p)$ is computed from $(\alpha, \beta)$ by

$$1 < a = 1 + \frac{1}{\alpha} < p = \frac{1+\alpha+\beta}{\alpha}, \tag{5}$$

and $\widetilde{f} \in C^{\infty}(\mathbb{R}_+)$ satisfies

$$\widetilde{f}(u) = o(u^p) \quad \text{and} \quad \widetilde{f}'(u) = o(u^{p-1}) \quad \text{as} \quad u \to +\infty. \tag{6}$$

If $\Omega = \mathbb{R}^N$ we have certain extra assumptions on $f$ and $u$.

- Conversely, given $1 < a < p < \infty$ for (4), we recover by $a = 1 + \frac{1}{\alpha}$ and $p = \frac{1+\alpha+\beta}{\alpha}$ the exponent $\beta$ for (1) and the transformation index $\alpha$ in (3). If we use, instead of $\beta$, the $p$ and $a$ notations to denote objects for $h$, we mean $\alpha = 1$.

# Blowup Profile

- In the non-perturbed case of (4) ($a = \widetilde{f} = 0$), namely, for the equation
$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1}u, \tag{7}$$
where $u(\cdot, t) : \mathbb{R}^N \to \mathbb{R}$, $p > 1$ and $p < \frac{N+2}{N-2}$ if $N \geq 3$, [Merle-Zaag Duke 1997] constructed a blowup solution with the following asymptotic behaviour
$$\left\| (T-t)^{\frac{1}{p-1}} u(\cdot, t) - \Phi_0 \left( \frac{\cdot - x_0}{\sqrt{(T-t)|\log(T-t)|}} \right) \right\|_{L^\infty(\Omega)} \longrightarrow 0 \quad \text{as} \quad t \to T, \tag{8}$$
is justified, where $x_0$ is the only blowup point, $T$ is the blowup time, and
$$\Phi_0(z) = \frac{1}{\left( p - 1 + \frac{(p-1)^2}{4p} |z|^2 \right)^{\frac{1}{p-1}}}. \tag{9}$$
Let $\kappa = \Phi_0(0) = (p-1)^{-\frac{1}{p-1}}$. The so-called final blowup profile for (7)
$$\lim_{t \to T} u(x,t) \quad \sim \quad \left[ \frac{(p-1)^2}{8p} \frac{|x - x_0|^2}{|\log|x - x_0||} \right]^{-\frac{1}{p-1}} \quad \text{as} \quad x \to x_0,$$
was later derived in [Zaag AIHP 1998].
- The constructive method in [Merle-Zaag] is robust and in fact has been extended by Zaag and collaborators to a considerably large class of parabolic problems.

# Intermediate and Final Profiles

- Introduce the intermediate profile

$$\widehat{\Phi}(z) = \left(\beta + 1 + \frac{(\beta+1)^2}{4\beta}|z|^2\right)^{\frac{1}{\beta+1}}, \quad z \in \mathbb{R}^N, \tag{10}$$

with the intermediate gradient profile being

$$\nabla\widehat{\Phi}(z) = \frac{\beta+1}{2\beta} z \left(\beta + 1 + \frac{(\beta+1)^2}{4\beta}|z|^2\right)^{-\frac{\beta}{\beta+1}}, \quad z \in \mathbb{R}^N. \tag{11}$$

Here $z$ will be a time-dependent rescaling of $x$ as in (8)-(9) above.

- Final profile (when $\Omega$ is bounded). Let $x_0 \in \Omega \subset \mathbb{R}^N$ and $\varrho_0 = \mathrm{dist}\,(x_0, \partial\Omega)$. Define

$$H^*_{x_0}(x) = \left[\frac{(\beta+1)^2}{8\beta}\frac{|x-x_0|^2}{|\log|x-x_0||}\right]^{\frac{1}{\beta+1}}, \quad 0 < |x-x_0| \le \min\left\{C(\beta), \frac{\varrho_0}{4}\right\}, \tag{12}$$

$$H^*_{x_0}(x) = 1, \quad |x-x_0| \ge \frac{\varrho_0}{2},$$

with the additional requirements that

$$\forall\ x \neq x_0, \quad H^*_{x_0}(x) > 0 \quad \text{and} \quad |\nabla H^*_{x_0}(x)| > 0.$$

For $\beta = 1$, the (vortex line) profile $H^*(x)$ in (12) is nearly straight (up to a logarithm $\log|x|$). For $\beta > 1$, the profile forms a cusp at $x = 0$.

The gradient of $H^* = H_0^*$ satisfies that as $x \to 0$,

$$\nabla H^*(x) \quad \sim \quad \frac{1}{\sqrt{2\beta}} \frac{x}{|x|} \frac{1}{\sqrt{|\log|x||}} \left[ \frac{(\beta+1)^2}{8\beta} \frac{|x|^2}{|\log|x||} \right]^{\frac{1}{\beta+1} - \frac{1}{2}}. \tag{13}$$

When $|x| \to 0$, $|\nabla H^*(x)|$ blows up if $\beta > 1$ and extinguishes if $0 < \beta \le 1$.

If $\Omega$ is a bounded domain in $\mathbb{R}^N$, then we set

$$\mathbf{H} = \left\{ \mathbf{h} \in W^{1,\infty}(\Omega) : \frac{1}{\mathbf{h}} \in L^\infty(\Omega) \right\}. \tag{14}$$

## Existence (& Stability) and Profiles

### Theorem (Merle-Zaag, 1997)

*Assume that* $\Omega$ *is bounded. For all* $x_0 \in \Omega$, *there exists a positive* $h_0 \in \mathbf{H}$ *such that for a* $T > 0$, *equation* (1) *with initial data* $h_0$ *has a unique solution* $h(\cdot, t)$ *on* $[0, T)$ *satisfying*

$$\lim_{t \to T} h(x_0, t) = 0.$$

*Furthermore, (i) (Intermediate extinction profile)*

$$\lim_{t \to T} \left\| \frac{(T-t)^{\frac{1}{\beta+1}}}{h(\cdot, t)} - \frac{1}{\widehat{\Phi}(z_{x_0}(\cdot, t))} \right\|_{L^\infty(\Omega)} = 0, \tag{15}$$

*where* $\widehat{\Phi}$ *is given in* (10) *and*

$$z_{x_0}(x, t) = \frac{x - x_0}{\sqrt{(T-t)|\log(T-t)|}}.$$

*(ii) (Final extinction profile)* $h^*(x) := \lim_{t \to T} h(x, t)$ *exists for all* $x \in \Omega$ *and*

$$h^*(x) \sim H^*_{x_0}(x) \quad \text{as} \quad x \to x_0,$$

*where* $H^*_{x_0}$ *is given precisely in* (12).

# Refined Asymptotics and Final Gradient Profile

## Theorem (H.-Zaag, 2024)

*Assume that* $\Omega$ *is bounded. For all* $x_0 \in \Omega$*, there exists a positive* $h_0 \in \mathbf{H}$ *such that for a* $T = T(h_0) \in (0, e^{-1})$*, equation* (1) *with initial data* $h_0$ *has a unique solution* $h(\cdot, t)$ *on* $[0, T)$ *satisfying* $\lim_{t \to T} h(x_0, t) = 0$*. Furthermore, for this* $h$ *and for all* $t \in [0, T)$*:*

$$\left\| \frac{(T-t)^{\frac{1}{\beta+1}}}{h(\cdot, t)} - \frac{1}{\widehat{\Phi}(z_{x_0}(\cdot, t))} \right\|_{L^\infty(\Omega)} \le C \frac{\log(|\log(T-t)|)}{|\log(T-t)|} \tag{16}$$

*and for each* $K > 0$ *and* $\Omega_{t,K} = \left\{ x \in \Omega : |x - x_0| \le K\sqrt{(T-t)|\log(T-t)|} \right\}$,

$$\left\| (T-t)^{-\frac{1}{\beta+1}+\frac{1}{2}} \nabla h(\cdot, t) - \frac{(\nabla \widehat{\Phi})(z_{x_0}(\cdot, t))}{|\log(T-t)|^{\frac{1}{2}}} \right\|_{L^\infty(\Omega_{t,K})} \le C(K) \frac{\log(|\log(T-t)|)}{|\log(T-t)|}. \tag{17}$$

*Here,* $\widehat{\Phi}$ *is given in* (10) *and* $z_{x_0}(x, t)$ *is given in Merle-Zaag Theorem. Moreover:* $(\nabla h)^*(x) := \lim_{t \to T} \nabla h(x, t)$ *exists for all* $x \in \Omega \backslash \{x_0\}$*, and*

$$(\nabla h)^*(x) \sim \nabla H^*_{x_0}(x) \quad as \quad x \to x_0. \tag{18}$$

# First Rescaling Transformation

As in Giga-Kohn and Merle-Zaag, we introduce the transform

$$w(y,s) = w_T(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \tag{19}$$

where $(y,s)$ are the so-called *similarity variables* defined by

$$y = \frac{x}{\sqrt{T-t}} \quad \text{and} \quad s = -\log(T-t).$$

Here $s > 0$ when $T < 1$. The equation satisfied by $w$ is then

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} - a\frac{|\nabla w|^2}{w} + w^p + e^{-\frac{ps}{p-1}} \widetilde{f}(e^{\frac{s}{p-1}} w), \tag{20}$$

where $\widetilde{f}$ is in (4). Moreover, (15) for $x_0 = 0$ is then equivalent to

$$\lim_{s\to\infty} \left\| w(\cdot,s) - \Phi\left(\frac{\cdot}{\sqrt{s}}\right) \right\|_{L^\infty(\mathbb{R}^N)} = 0, \tag{21}$$

where

$$\Phi(z) = \Phi_a(z) = \left(p - 1 + \frac{(p-1)^2}{4(p-a)}|z|^2\right)^{-\frac{1}{p-1}}. \tag{22}$$

See [Merle-Zaag, Remark 1.1] for the formal derivation of the profile $\Phi_0$.

To better describe (21), we introduce

$$q = w - \varphi, \quad \varphi(y,s) = \Phi\left(\frac{y}{\sqrt{s}}\right) + \frac{N\kappa}{2(p-a)s}, \tag{23}$$

and one adds $\frac{N\kappa}{2(p-a)s}$ to simplify the calculations. Observe that $q$ satisfies

$$\frac{\partial q}{\partial s} = \mathcal{L}_V(q) + B(q) + T(q) + R + L(q), \tag{24}$$

where

$$\begin{aligned}
\mathcal{L}_V &= \mathcal{L} + V, \quad \mathcal{L} = \Delta - \frac{1}{2} y\cdot\nabla + 1,\\
V &= p\left(\varphi^{p-1} - \kappa^{p-1}\right), \quad \kappa = \frac{1}{(p-1)^{\frac{1}{p-1}}},\\
B(q) &= (\varphi+q)^p - \varphi^p - p\varphi^{p-1}q,\\
T(q) &= -a\frac{|\nabla(\varphi+q)|^2}{\varphi+q} + a\frac{|\nabla\varphi|^2}{\varphi},\\
R &= -\frac{\partial\varphi}{\partial s} + \Delta\varphi - \frac{1}{2}y\cdot\nabla\varphi - \frac{\varphi}{p-1} - a\frac{|\nabla\varphi|^2}{\varphi} + \varphi^p,\\
L(q) &= e^{-\frac{ps}{p-1}}\widetilde{f}\left(e^{\frac{s}{p-1}}(\varphi+q)\right).
\end{aligned} \tag{25}$$

Here $\widetilde{f}$ is given in (4). In $B(q)$, $T(q)$ and $L(q)$, note that $\varphi + q = w > 0$.

# Linear operator $\mathcal{L}$

Consider the Hilbert space with Gaussian measure

$$L^2_\rho = L^2(\mathbb{R}^N, \rho dy), \quad \text{where } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{N}{2}}}.$$

The operator $\mathcal{L} = \Delta - \frac{1}{2} y \cdot \nabla + 1$ that appears in (24) is self-adjoint in $L^2_\rho$. Moreover, it has explicit spectrum as follows

$$\text{Spec}(\mathcal{L}) = \left\{ \lambda_m = 1 - \frac{m}{2} \middle| m \in \mathbb{N} \right\}. \tag{26}$$

Corresponding to the eigenvalue $\lambda_m$, we have the eigenspace $\mathcal{E}_m$ given by

$$\mathcal{E}_m = \text{Span}\left\{ h_{m_1}(y_1) h_{m_2}(y_2) \cdots h_{m_N}(y_N) | m_1 + m_2 + \cdots + m_N = m \right\}, \tag{27}$$

where $h_\ell$ is the (rescaled) Hermite polynomial in one dimension, defined by

$$h_\ell(\xi) = \sum_{j=0}^{[\ell/2]} (-1)^j \frac{\ell!}{j!(\ell - 2j)!} \xi^{\ell - 2j},$$

and the first three such polynomials are 1, $\xi$ and $\xi^2 - 2$.

We introduce the following decomposition, for any $r \in L^{\infty}(\mathbb{R}^N)$:

$$r(y) = \chi(y,s)r(y) + (1-\chi(y,s))r(y) =: r_b(y,s) + r_e(y,s), \tag{28}$$

where

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K_0\sqrt{s}}\right), \tag{29}$$

$\chi_0$ being a one-dimensional cut-off satisfying

$$\operatorname{supp}\chi_0 \subset [0,2], \quad 0 \le \chi_0 \le 1 \quad \text{and} \quad \chi_0 \equiv 1 \quad \text{on} \quad [0,1]. \tag{30}$$

We refer to (28) as *inner-outer decomposition* since

$$\operatorname{supp} r_b(s) \subset \{|y| \le 2K_0\sqrt{s}\} \quad \text{and} \quad \operatorname{supp} r_e(s) \subset \{|y| \ge K_0\sqrt{s}\}. \tag{31}$$

At the blowup point $x = 0$, $(0,t)$ for $t < T$ is always inside (resp. outside) the support of the first (resp. second) function. The subscripts mean "blowup" and "exterior".

Next we note that the set of eigenfunctions of $\mathcal{L}$ makes a basis of $L^2_\rho$.
We write $r_b \in L^2_\rho$ into the following spectral decomposition

$$r_b(y,s) = r_0(s) + r_1(s)\cdot y + y^T \cdot r_2(s)\cdot y - 2\mathrm{Tr}(r_2(s)) + r_-(y,s), \tag{32}$$

where

$$r_m(s) = \{P_\beta[r_b(s)]\}_{\beta\in\mathbb{N}^N,\ |\beta|=m},\ \ m\geq 0, \tag{33}$$

with $P_\beta[r_b]$ being the projection of $r_b$ on the eigenfunction $h_\beta$

$$P_\beta[r_b(s)] = \int_{\mathbb{R}^N} r_b(y,s)\frac{h_\beta}{\|h_\beta\|^2_{L^2_\rho}}\rho(y)dy, \tag{34}$$

and

$$r_-(y,s) = P_-[r_b(s)] = \sum_{\beta\in\mathbb{N}^N,\ |\beta|\geq 3} P_\beta[r_b(s)]h_\beta(y). \tag{35}$$

Throughout the talk we shall use the following expansion

$$r(y) = r_0(s) + r_1(s)\cdot y + y^T\cdot r_2(s)\cdot y - 2\mathrm{Tr}(r_2(s)) + r_-(y,s) + r_e(y,s). \tag{36}$$

Note that $L^\infty(\mathbb{R}^N) \subset L^2_\rho$.

**Definition**. For $T \in (0,1)$, $t \in (0,T)$ and $x \in \Omega$, let

$$s = -\log(T-t) > 0, \quad y = \frac{x}{\sqrt{T-t}} \quad \text{and} \quad z = \frac{y}{\sqrt{s}} = \frac{x}{\sqrt{(T-t)|\log(T-t)|}}.$$

We define, for $K_0 > 0$, $\varepsilon_0 > 0$ and $t \in (0,T)$ given, three regions of $x$ that cover $\Omega$:

$$\begin{aligned}\mathcal{R}_1(K_0,\varepsilon_0,t) &= \left\{x \in \Omega \middle| |x| \le K_0\sqrt{(T-t)|\log(T-t)|}\right\} \\ &= \{x \in \Omega | |y| \le K_0\sqrt{s}\} = \{x \in \Omega | |z| \le K_0\},\end{aligned} \tag{37}$$

$$\begin{aligned}\mathcal{R}_2(K_0,\varepsilon_0,t) &= \left\{x \in \Omega \middle| \frac{K_0}{4}\sqrt{(T-t)|\log(T-t)|} \le |x| \le \varepsilon_0\right\} \\ &= \left\{x \in \Omega \middle| \frac{K_0}{4}\sqrt{s} \le |y| \le \varepsilon_0 e^{\frac{s}{2}}\right\} = \left\{x \in \Omega \middle| \frac{K_0}{4} \le |z| \le \varepsilon_0 \frac{e^{\frac{s}{2}}}{\sqrt{s}}\right\},\end{aligned} \tag{38}$$

$$\begin{aligned}\mathcal{R}_3(K_0,\varepsilon_0,t) &= \left\{x \in \Omega \middle| |x| \ge \frac{\varepsilon_0}{4}\right\} \\ &= \left\{x \in \Omega \middle| |y| \ge \frac{\varepsilon_0}{4} e^{\frac{s}{2}}\right\} = \left\{x \in \Omega \middle| |z| \ge \frac{\varepsilon_0}{4}\frac{e^{\frac{s}{2}}}{\sqrt{s}}\right\}.\end{aligned} \tag{39}$$

Set $\mathcal{R}_i = \mathcal{R}_i(K_0,\varepsilon_0) = \{(x,t) \in \Omega \times (0,T) | x \in \mathcal{R}_i(K_0,\varepsilon_0,t)\}$.

We shall localize the study of (1.1) away from the extinction point. To this aim, introduce

$$k_x(\xi,\tau) = \frac{h\left(x + \sqrt{T - t(x)}\xi, t(x) + (T - t(x))\tau\right)}{(T - t(x))^{\frac{1}{\beta+1}}}, \tag{40}$$

where $x \in \Omega\backslash\{0\}$ and $t(x) < T$ is determined by the quasi-parabola

$$\frac{K_0}{4}\sqrt{(T - t(x))|\log(T - t(x))|} = |x|, \tag{41}$$

which is exactly the relation giving the inner boundary of the region (38). For the sake of convenience, we denote the radial function (abusing a little bit the notation)

$$\theta(x) = T - t(x) = \theta(|x|), \tag{42}$$

hence

$$t(x) \to T, \quad \theta(x) \to 0 \quad \text{and} \quad |\log\theta(x)| \to +\infty, \quad \text{as } x \to 0.$$

Moreover, for $t(x) \geq 0$, by a simple calculation $T < e^{-1}$ implies that $|x|$ is bounded, in this sense our localization is also away from the boundary of $\Omega$.

Definition. Let $T \in (0, e^{-1})$. Fix $\underline{\alpha} > 3$ and $\underline{\alpha} + 1 \leq \overline{\alpha} < \infty$. Let $K_0 > 0$, $\varepsilon_0 > 0$, $A > 0$, $\alpha_0 > 0$, $\delta_0 > 0$, $C_0 > 0$ and $\eta_0 > 0$.
(I) For all $t_0 \in [0, T)$ and for all $t \in [t_0, T)$, we define

$$S^*(t_0, t) = S^*(t_0, K_0, \varepsilon_0, A, \alpha_0, \delta_0, C_0, \eta_0, t)$$

as the set of functions $\mathbf{h} \in \mathbf{H}$, where $\mathbf{H}$ is defined in (14), satisfying
(i) Estimates in $\mathcal{R}_1(K_0, \varepsilon_0, t)$: We require, with $s = -\log(T - t) > 1$, that

$$q(\cdot, s) \in V_{K_0,A}(s),$$

with $q(\cdot, s) = w(\cdot, s) - \varphi(\cdot, s)$ defined in (23) through transforming $h(\cdot, t) := \mathbf{h}(\cdot)$ first into $u(\cdot, t)$ via (3) and then into $w(\cdot, s)$ via (19), and $V_{K_0,A}(s)$ is the set of functions $\mathbf{r} \in L^\infty(\mathbb{R}^N)$ such that $r(\cdot, s) := \mathbf{r}(\cdot)$ satisfies

$$\begin{aligned} &(r_0(s), r_1(s)) \in \left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^{1+N} =: \mathcal{Q}_A(s), \quad |r_2(s)| \leq \frac{A^2 \log s}{s^2}, \\ &\left\| \frac{r_-(\cdot, s)}{1 + |\cdot|^3} \right\|_{L^\infty} \leq \frac{A^{\underline{\alpha}} \log s}{s^{\frac{5}{2}}}, \quad \|r_e(\cdot, s)\|_{L^\infty} \leq \frac{A^{\overline{\alpha}} \log s}{s}, \end{aligned} \tag{43}$$

where $r_m(s)$, $m \in \{0, 1, 2\}$, $r_-(\cdot, s)$ and $r_e(\cdot, s)$ are components of $r(\cdot, s)$ in the inner-outer+spectral decompositions associated to $s$ and $K_0$ in defining cut-off $\chi_0$.

(ii) Estimates in $\mathcal{R}_2(K_0, \varepsilon_0, t)$: For all

$$t \in [t_0, T) \quad \text{and} \quad |x| \in \left[\frac{K_0}{4}\sqrt{(T-t)|\log(T-t)|}, \varepsilon_0\right],$$

and for all

$$|\xi| \le \alpha_0 \sqrt{|\log \theta(x)|} \quad \text{and} \quad \tau \in \left[\frac{t_0 - t(x)}{T - t(x)}, 1\right),$$

we require

$$\left|k_x(\xi,\tau) - \widehat{k}(\tau)\right| \le \delta_0, \tag{44}$$

$$|\nabla_\xi k_x(\xi,\tau)| \le \frac{C_0}{\sqrt{|\log\theta(x)|}}. \tag{45}$$

Here, $k_x$ is defined in (40) via $h(\cdot, t) := \mathbf{h}(\cdot)$,

$$\tau = \tau(x,t) = \frac{t - t(x)}{\theta(x)}, \quad \theta(x) = T - t(x),$$

where $t(x)$ is given in (41) and $\widehat{k}$ solves

$$\frac{d\widehat{k}}{d\tau} = -\frac{1}{\widehat{k}^\beta}.$$

(iii) Estimates in $\mathcal{R}_3(K_0, \varepsilon_0, t)$: For all $|x| \geq \frac{\varepsilon_0}{4}$, we require on $h(\cdot, t) := \mathbf{h}(\cdot)$ that

$$|h(x,t) - h(x,t_0)| \leq \eta_0, \tag{46}$$

$$|\nabla h(x,t) - \nabla h(x,t_0)| \leq \eta_0. \tag{47}$$

(II) For all $t_0 \in [0, T)$, we define

$$\begin{aligned} S^*(t_0) &= S^*(t_0, K_0, \varepsilon_0, A, \alpha_0, \delta_0, C_0, \eta_0) \\ &= \Big\{ h \in C([t_0, T); \mathbf{H}) : \\ &\qquad \forall t \in [t_0, T), h(\cdot, t) \in S^*(t_0, K_0, \varepsilon_0, A, \alpha_0, \delta_0, C_0, \eta_0, t) \Big\}. \end{aligned}$$

• The involved parameters for $S^*(t_0, K_0, \varepsilon_0, A, \alpha_0, \delta_0, C_0, \eta_0)$ are arranged in the order of their appearance in this definition. It is clear that the set $V_{K_0,A}(s)$ is shrinking to 0 (respectively, $S^*(t_0, t)$ to the extinction profile) when $s \to \infty$ (respectively, $t \to T$). Recall that our aim is to find a solution $q$ of (24) with $\|q(s)\|_{L^\infty} \to 0$.

# Initial Data

Let us consider initial data $h(t_0)$ in the following form: for $(d_0, d_1) \in \mathbb{R}^{1+N}$, define

$$h(x, t_0; d_0, d_1)$$
$$:= (T - t_0)^{\frac{1}{\beta+1}} \alpha^{\frac{1}{\beta+1}} \left[ \Phi(z) + (d_0 + d_1 \cdot z)\chi_0 \left( \frac{|z|}{K_0/16} \right) \right]^{-\frac{1}{\alpha}} \Bigg|_{t=t_0} \chi_1(x, t_0) \quad (48)$$
$$+ H^*(x)\,(1 - \chi_1(x, t_0)),$$

where

$$\chi_1(x, t_0) = \chi_0 \left( \frac{|x|}{(T - t_0)^{\frac{1}{2}} |\log(T - t_0)|^{\frac{p}{2}}} \right), \quad (49)$$

$H^*(x)$, $(z, \Phi(z))$ and $\chi_0$ are recalled as before.

In particular, for $z|_{t=t_0}$ large, the initial data $h(t_0)$ agrees with $H^*(x)$, while on $\{\chi_1(\cdot, t_0) = 1\}$, $h(t_0)$ and its transformation $u(t_0)$ are well-prepared around $\Phi(z)|_{t=t_0}$.

The roadmap which finally leads to the proof of the Main Theorem consists of six parts.

**Part I. Initialization of the Evolution Problem.**

**Part II. Parabolic Regularity under the Partial Trapping Assumption.**

**Part III. A Priori Estimates in $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$.**

**Part IV. Finite Dimensional Reduction.**

**Part V. Contradiction via Topological Arguments.**

**Part VI. Full Trapping Implies the Gradient Profile.**

The complete proof (with full details) is given in [H.-Zaag J. Evol. Equ. 2024].

One technical part in above roadmap is the *A Priori* Estimates in $\mathcal{R}_1$ ("self-improving"). It concerns the "mode dynamics" of error $q$ in (24) for $h$ trapped in the shrinking set.

Here is the statement for the negative spectrum part and the exterior part.
For $v_m(s)$, $m \in \{0,1,2\}$, $v_-(\cdot, s)$ and $v_e(\cdot, s)$ the components of $v(\cdot, s)$ in the inner-outer and spectral decompositions associated to $s$ and $K_0$ in defining cut-off $\chi_0$, satisfying

$$\sum_{m=0}^{2} |v_m(\sigma)| + \left\| \frac{v_-(\cdot,\sigma)}{1+|\cdot|^3} \right\|_{L^\infty} + \|v_e(\sigma)\|_{L^\infty} < \infty,$$

one has

$$\left\| \frac{\mathbb{V}_-(\cdot,s)}{1+|\cdot|^3} \right\|_{L^\infty} \le C \frac{e^{s-\sigma}((s-\sigma)^2+1)}{s} \left(|v_0(\sigma)| + |v_1(\sigma)| + \sqrt{s}|v_2(\sigma)|\right) + Ce^{-\frac{s-\sigma}{2}} \left\| \frac{v_-(\cdot,\sigma)}{1+|\cdot|^3} \right\|_{L^\infty} + C\frac{e^{-(s-\sigma)^2}}{s^{\frac{3}{2}}} \|v_e(\sigma)\|_{L^\infty},$$

and

$$\|\mathbb{V}_e(s)\|_{L^\infty} \le Ce^{s-\sigma} \left( \sum_{m=0}^{2} s^{\frac{m}{2}} |v_m(\sigma)| + s^{\frac{3}{2}} \left\| \frac{v_-(\cdot,\sigma)}{1+|\cdot|^3} \right\|_{L^\infty} \right) + Ce^{-\frac{s-\sigma}{p}} \|v_e(\sigma)\|_{L^\infty},$$

where $\mathbb{V}(s) = \mathcal{K}(s,\sigma)v(\sigma)$ and $\mathcal{K}$ is the fundamental solution associated to $\partial_s q = \mathcal{L}_V q$.
The behavior of the kernel $\mathcal{K}$ follows from a perturbation method around Mehler Kernel.

$$e^{\theta\mathcal{L}}(y,x) = \frac{e^{\theta}}{(4\pi(1-e^{-\theta}))^{\frac{N}{2}}} \exp\left[-\frac{|ye^{-\frac{\theta}{2}} - x|^2}{4(1-e^{-\theta})}\right].$$

Thanks for your kind attention!

Enjoy your trip in Hangzhou!