Lower estimates and Hölder regularity of the heat kernels for non-local Dirichlet form on doubling spaces

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https://www.math.uni-bielefeld.de/~grigor/tp-le.pdf

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2. Stable-like type of heat kernel estimates
3. Main results
   - Near-diagonal lower estimates
   - Off-diagonal lower estimates
   - Main ingredients of proofs
4. Example

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Heat kernel

Let (M, d, μ) be a metric measure space, i.e., (M, d) is a locally compact separable metric space and μ is a Radon measure on (M, d) with full support.

- Let $(\mathcal{E}, \mathcal{F})$ be a regular pure jump type Dirichlet form on $L^2(M)$;
- Let $\mathcal{L}$ be the generator of $(\mathcal{E}, \mathcal{F})$. That is, $\mathcal{L}$ is self-adjoint, negative definite operator;
- $\{P_t\}$ be the heat semigroup associated with $(\mathcal{E}, \mathcal{F})$: $P_t = e^{\mathcal{L}t}$.

If for every $t > 0$, the operator P_t has an *integral kernel* $p_t(x, y)$:

$$P_t f(\cdot) = \int_M p(t, \cdot, y) f(y) dy, \quad t > 0,\ f \in L^2(M),$$

we say the function $p_t(x, y)$ is the *heat kernel* of $\mathcal{L}$, or $\{P_t\}$, or $(\mathcal{E}, \mathcal{F})$.

In this case, $p_t(x, y)$ is the transition density of the pure jump process generated by $\mathcal{L}$.

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Heat kernel estimate

In this talk, we are mainly concerned with equivalent characterizations of heat kernel lower bounds for some “singular” non-local Dirichlet forms on metric measure spaces. We focus on pure jump type Dirichlet forms.

Regular jump type Dirichlet forms

By [1, Theorem 3.2.1 on p.120], any pure jump type Dirichlet form $(\mathcal{E}, \mathcal{F})$ admits the following expression:

$$\mathcal{E}(u,v) = \iint_{M\times M\setminus\mathrm{diag}} (u(x)-u(y))(v(x)-v(y))dj(x,y),$$

where j is a Radon measure (called jump measure) defined on $M \times M\setminus\mathrm{diag}$, and the double integral "$\iint_{A\times B} f(x,y)dj(x,y)$" means that the variable x belongs to A and y belongs to B.

[1] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2011.

Examples of singular jump type Dirichlet forms

Example 1. Let $\{X_t^{(i)}, 1 \le i \le d\}$ be d independent copies of one-dimensional symmetric stable process of index $\beta \in (0,2)$. Then, the process

$$X_t := (X_t^{(1)}, X_t^{(2)}, \cdots, X_t^{(d)})$$

- is a pure jump process on $\mathbb{R}^d$
- is a highly singular process with jump measure:

$$\begin{aligned} dj(x,y) &= J(x,dy)dx \\ &:= \sum_{i=1}^{d} d\delta_{x^{(1)}}(y^{(1)}) \cdots d\delta_{x^{(i-1)}}(y^{(i-1)}) \frac{C_{1,\beta}}{|x^{(i)} - y^{(i)}|^{1+\beta}}\, dy^{(i)} \\ &\quad \cdot d\delta_{x^{(i+1)}}(y^{(i+1)}) \cdots d\delta_{x^{(d)}}(y^{(d)})\, dx, \end{aligned}$$

where $x = (x^{(1)}, x^{(2)}, \cdots, x^{(d)})$, $y = (y^{(1)}, y^{(2)}, \cdots, y^{(d)}) \in \mathbb{R}^d$.

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where $x = (x^{(1)}, x^{(2)}, \cdots, x^{(d)})$, $y = (y^{(1)}, y^{(2)}, \cdots, y^{(d)}) \in \mathbb{R}^d$.

Example 1 - continue

The kernel $J(x, dy)$ can be interpreted as follows:

$$\int_{\mathbb{R}^d} f(y)J(x, dy) = \sum_{i=1}^{d} \text{P.V. } C_{1,\beta} \int_{\mathbb{R}} \frac{f(x^{(1)}, \cdots, x^{(i-1)}, \rho, x^{(i+1)}, \cdots, x^{(d)})}{|x^{(i)} - \rho|^{1+\beta}} d\rho.$$

The process X_t is the *cylindrical stable process* on $\mathbb{R}^d$.

The jump measure $dj(x, y) = J(x, dy)dx$ determines a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $\mathbb{R}^d$ with an appropriate domain $\mathcal{F} \subset L^2(\mathbb{R}^d)$.

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Example 2

Consider the truncated fractional Lapacian $\mathcal{L} := \overline{\Delta}^{\frac{\beta}{2}}$ with $\beta \in (0,2)$ on $\mathbb{R}^d$:

$$\overline{\Delta}^{\frac{\beta}{2}} f(x) = \text{ P.V. } C_{d,\beta} \int_{\mathbb{R}^d} \frac{f(y) - f(x)}{|x-y|^{d+\beta}} \mathbf{1}_{\{|x-y| \le 1\}} dy, \quad f \in C_0^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

The associated Dirichlet form has the jump kernel:

$$J(x,y) := \frac{dj(x,y)}{dxdy} = \frac{C_{d,\beta}}{|x-y|^{d+\beta}} \mathbf{1}_{\{|x-y| \le 1\}}.$$

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In these examples, either the jump measure j is NOT absolutely continuous with respect to $\mu \times \mu$, or the jump kernel may vanish in some areas. The associated Dirichlet forms are *singular*.

There are more examples in the paper:

[1] B. Dyda and M. Kassmann. Regularity estimates for elliptic nonlocal operators. *Anal. PDE.* 13:317 – 370, 2020.

There are few results on heat kernels for singular Dirichlet form on metric measure spaces.

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There are few results on heat kernels for singular Dirichlet form on metric measure spaces.

What type of lower bounds do the heat kernels of those examples have?

- In Example 1, the heat kernel $p_t(x, y)$ of the cylindrical stable process exists and satisfies:

$$p_t(x,y) \geq ct^{-\frac{d}{\beta}}, \quad \text{for } x, y \in \mathbb{R}^d \text{ with } |x-y| \leq t^{1/\beta}; \tag{1}$$

- In Example 2, the heat kernel $p_t(x, y)$ of the truncated β-stable process exists and also satisfies (1);
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Fix $\alpha > 0$ and $\beta > 0$. Let $(\mathcal{E}, \mathcal{F})$ be a regular pure jump type Dirichlet form on (M, d, μ) associated with the jump measure j satisfying $dj(x,y) = J(x,y)d\mu(x)d\mu(x)$ and

$$J(x,y) \asymp d(x,y)^{-(\alpha+\beta)}, \quad x,y \in M. \tag{J}$$

In this case, $(\mathcal{E}, \mathcal{F})$ is called *stable-like*.

We use the notation:

$$\mu(B(x,r)) \asymp r^{\alpha}, \quad x \in M,\ r > 0. \tag{V}$$

For any $t > 0$ and $\mu \times \mu$-almost all $(x,y) \in M \times M$,

$$p_t(x,y) \asymp \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{UE) + (LE}$$

Note that (UE) + (LE) are equivalent to

$$p_t(x,y) \asymp \begin{cases} \frac{1}{t^{\alpha/\beta}}, & \text{if } d(x,y) < t^{1/\beta}, \\ \frac{t}{d(x,y)^{\alpha+\beta}}, & \text{if } d(x,y) \geq t^{1/\beta}. \end{cases}$$

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Known results

In the case $\beta < 2$, Chen and Kumagai proved in [1] that under the volume condition (V), the following equivalence holds true:

$$(J) \Leftrightarrow (UE) + (LE).$$

Moreover, the heat kernel $p_t(x, y)$ is Hölder continuous.

For general $\beta > 0$, the heat kernel of $(\mathcal{E}, \mathcal{F})$ has been studied independently by Chen, Kumagai and Wang [2], and Grigor'yan, E. Hu and Jiaxin Hu [3].

[1] Z.-Q. Chen and T. Kumagai. Heat kernel estimates for stable-like processes on d -sets. *Stochastic Process. Appl.* 108(1):27–62, 2003.

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Theorem

Let $(\mathcal{E}, \mathcal{F})$ be a regular, pure jump type Dirichlet form on $L^2(M, \mu)$ with a jump kernel J. If (V) is satisfied, then

$$(\text{Gcap}) + (\text{J}) \Leftrightarrow (\text{UE}) + (\text{LE}).$$

Moreover, under these hypotheses, the heat kernel $p_t(x, y)$ is Hölder continuous jointly in x, y.

Note that in the case $\beta < 2$, under (V),

$$(\text{J}) \Leftrightarrow (\text{UE}) + (\text{LE}).$$

We will introduce the *generalized capacity condition* (Gcap) later in a more general setting.

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Let (M, d, μ) be a metric measure space.
Let $V(x,R) := \mu(B(x,R))$, where $B(x,R) := \{y \in M : d(y,x) < R\}$.

Definition

We say that condition (VD) holds if there exists $C > 0$ such that for all $x \in M$ and $R > 0$,

$$V(x, 2R) \leq CV(x,R),$$

Note that condition (VD) is equivalent to the following: there exists $\alpha > 0$ such that, for all $x, y \in M$ and all $0 < r \leq R$,

$$\frac{V(x,R)}{V(y,r)} \leq C\left(\frac{d(x,y)+R}{r}\right)^{\alpha}.$$

Definition

We say that condition (RVD) holds if there exist $C, \alpha' > 0$ such that for all $x \in M$ and $0 < r \leq R$,

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The scaling function W

Recall the stable-like case on $\mathbb{R}^d$.

1. $V(x,r) \asymp r^d$.

2. The jump kernel $J(x,y)$ exists, that is,

$$dj(x,y) = J(x,y)d\mu(x)d\mu(y).$$

And, $J(x,y) \asymp d(x,y)^{-(d+\beta)} = \frac{1}{d(x,y)^d} \cdot \frac{1}{d(x,y)^\beta}$.

$d(x,y)^d \asymp V(x,d(x,y))$,

$d(x,y)^\beta \to W(x,d(x,y))$ called *space/time scaling function.*

W describes how the process jumps in some sense.

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Let $W : M \times [0,\infty] \to [0,\infty]$ be a function such that for each $x \in M$, $W(x,\cdot)$ is continuous, strictly increasing, and $W(x,0) = 0$, $W(x,\infty) = \infty$. Assume also that there exist C, β_1, β_2 ($\beta_1 \le \beta_2$) such that, for all $0 < r \le R < \infty$ and for all $x, y \in M$ with $d(x,y) \le R$,

$$C^{-1}\left(\frac{R}{r}\right)^{\beta_1} \le \frac{W(x,R)}{W(y,r)} \le C\left(\frac{R}{r}\right)^{\beta_2}. \tag{2}$$

Clearly, we have by (2) that, for all $x \in M$ and all $0 < r \le R < \infty$

$$C^{-1}\left(\frac{R}{r}\right)^{1/\beta_2} \le \frac{W^{-1}(x,R)}{W^{-1}(x,r)} \le C\left(\frac{R}{r}\right)^{1/\beta_1},$$

where $W^{-1}(x,\cdot)$ is the inverse function of $W(x,\cdot)$ for $x \in M$.

The scaling function W

Let $W : M \times [0, \infty] \to [0, \infty]$ be a function such that for each $x \in M$, $W(x, \cdot)$ is continuous, strictly increasing, and $W(x, 0) = 0$, $W(x, \infty) = \infty$. Assume also that there exist C, β_1, β_2 ($\beta_1 \leq \beta_2$) such that, for all $0 < r \leq R < \infty$ and for all $x, y \in M$ with $d(x, y) \leq R$,

$$C^{-1} \left(\frac{R}{r} \right)^{\beta_1} \leq \frac{W(x, R)}{W(y, r)} \leq C \left(\frac{R}{r} \right)^{\beta_2}. \tag{2}$$

Clearly, we have by (2) that, for all $x \in M$ and all $0 < r \leq R < \infty$

$$C^{-1} \left(\frac{R}{r} \right)^{1/\beta_2} \leq \frac{W^{-1}(x, R)}{W^{-1}(x, r)} \leq C \left(\frac{R}{r} \right)^{1/\beta_1},$$

where $W^{-1}(x, \cdot)$ is the inverse function of $W(x, \cdot)$ for $x \in M$.

Examples of the function W

Let $\beta > \varepsilon > 0$, $\beta_1, \beta_2 > 0$.

- $W(x,r) \asymp r^\beta$;
- $W(x,r) \asymp r^\beta \log(r+2)$;
- $W(x,r) \asymp r^{\beta_1} + r^{\beta_2}$;
- $W(x,r) \asymp a(x) r^\beta$, where $a : M \mapsto \mathbb{R}$ satisfies:

$$0 < c_1 \leq a(x) \leq c_2, \quad x \in M;$$

- $W(x,r) = \left(\frac{|x|+r}{r}\right)^\varepsilon r^\beta$;
- $\cdots\cdots$

We will formulate conditions in terms of volume function V and space/time scaling function W.

1. Heat kernel and singular Dirichlet forms

2. Stable-like type of heat kernel estimates

3. Main results
   - Near-diagonal lower estimates
   - Off-diagonal lower estimates
   - Main ingredients of proofs

4. Example

To investigate the conditions for the lower bounds, we are motivated by the papers

[1] A. Bendikov, A. Grigor'yan, E. Hu, and J. Hu. Heat kernels and non-local Dirichlet forms on ultra-metric spaces. *Ann. Scuola Norm. Sup. Pisa*, 1:399–461, 2021.

[2] A. Grigor'yan, E. Hu, and J. Hu. Two-sided estimates of heat kernels of jump type Dirichlet forms. *Adv. Math.*, 330:433–515, 2018.

[3] Z.-Q. Chen, T. Kumagai, and J. Wang. Stability of heat kernel estimates for symmetric jump processes on metric measure spaces. To appear in *Memoirs Amer. Math. Soc.*.

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In [1], to cover the case when Dirichlet form is singular, the condition

$$J(x,y) \geq Cd(x,y)^{-(\alpha+\beta)},$$

was replaced by Poincaré inequality, and the condition

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Poincaré inequality

Let $\mathcal{F}' = \mathcal{F} + \{const\}$.

Definition

We say that the *Poincaré inequality* (PI) holds if there exist constants $C > 0$ and $\kappa \in (0, 1]$ such that, for any ball $B := B(x_0, R)$ and any function $u \in \mathcal{F}' \cap L^\infty$,

$$\int_{\kappa B} |u - u_{\kappa B}|^2 d\mu \le C W(x_0, R) \iint_{B \times B} (u(x) - u(y))^2 dj(x, y),$$

where u_A is the mean of the function u over A (that is, $u_A := \frac{1}{\mu(A)} \int_A u d\mu$).

Definition

We say that *condition* (TJ) is satisfied if there exists a kernel J on $M \times \mathcal{B}(M)$ such that $dj(x, y) = J(x, dy)d\mu(x)$ in $M \times M$, and for any $x \in M$ and any $R > 0$,

$$J(x, B(x, R)^c) \le \frac{C}{W(x, R)},$$

where $C > 0$ is independent of x, R.

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where $C > 0$ is independent of x, R.

Condition (Gcap)

Definition ($\overline{\kappa}$-cutoff functions)

Let $U \subset M$ be an open set, A be a measurable subset of U and $\overline{\kappa} \geq 1$. A $\overline{\kappa}$-*cutoff function* of the pair (A, U) is any function $\phi \in \mathcal{F}$ such that

- $0 \leq \phi \leq \overline{\kappa} \quad \mu$-a.e. in M;
- $\phi \geq 1 \quad \mu$-a.e. in A;
- $\phi = 0 \quad \mu$-a.e. in U^c.

We denote by $\overline{\kappa}$-cutoff(A, U) the collection of all $\overline{\kappa}$-cutoff functions of the pair (A, U).

Define

$$\text{cutoff}(A, U) := 1\text{-}\,\text{cutoff}(A, U).$$

Note that, for any $\overline{\kappa} \geq 1$,

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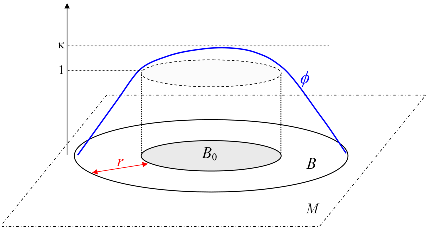
Condition (Gcap)

Definition

We say that the *generalized capacity condition* (Gcap) is satisfied if there exist two constants $\overline{\kappa} \geq 1, C > 0$ such that, for all $u \in \mathcal{F}' \cap L^\infty$ and all two concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $x_0 \in M$ and $0 < R < R + r$, there exists some $\phi \in \overline{\kappa}$- cutoff$(B_0, B)$ such that

$$\mathcal{E}(u^2\phi, \phi) \leq \sup_{x \in B} \frac{C}{W(x, r)} \int_B u^2 d\mu.$$

We remark that ϕ may depend on u, but $\overline{\kappa}, C$ are independent of u, B_0, B.



Main result-1: Near-diagonal lower estimate

Theorem

Let $(\mathcal{E}, \mathcal{F})$ be a regular, pure jump type Dirichlet form in L^2. If conditions (VD), (RVD), (TJ) are satisfied, then

$$(\mathrm{PI}) + (\mathrm{Gcap}) \Leftrightarrow (\mathrm{LLE}) \Rightarrow (\mathrm{NLE}).$$

(LLE): localized lower estimate

(NLE): Near-diagonal lower estimate

For any open set $\Omega \subset M$, let $\mathcal{F}(\Omega) := \overline{\mathcal{F} \cap C_c(\Omega)}^{\mathcal{E}_1}$, where $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2}$. Then $(\mathcal{E}, \mathcal{F}(\Omega))$ is also a regular Dirichlet form on $L^2(\Omega)$.

Definition (**Localized lower estimate**)

We say that *localized lower estimate* (LLE) if the followings are true.

1. For any bounded open set $\Omega \subset M$, the Dirichlet heat kernel $p_t^{\Omega}(x,y)$ exists and jointly continuous in $(x,y) \in \Omega \times \Omega$;
2. There exist $C > 0$ and $\delta \in (0,1)$ such that, for any ball $B := B(x_0, R)$ and for any $t \le W(x_0, \delta R)$,

$$p_t^B(x,y) \ge \frac{C}{V(x, W^{-1}(x,t))}, \quad x, y \in B(x_0, \delta W^{-1}(x_0,t)).$$

Definition (**Near-diagonal lower estimate**)

We say that *condition* (NLE) holds if $p_t(x,y)$ exists and satisfies a *near-diagonal lower estimate*: there exist two constants $\delta, C > 0$ such that

$$p_t(x,y) \ge \frac{C}{V(x, W^{-1}(x,t))}$$

for μ-almost all $x, y \in M$ and all $t > 0$ satisfying $d(x,y) < \delta W^{-1}(x,t)$.

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Theorem

Let $(\mathcal{E},\mathcal{F})$ *be a regular, pure jump type Dirichlet form in* L^2. *If conditions* (VD), (RVD), (TJ) *are satisfied, then*

$$(\text{PI}) + (\text{Gcap}) \Leftrightarrow (\text{LLE}) \Rightarrow (\text{NLE}).$$

Remark

One can construct a example (similar to cylindrical stable process) to show that under the assumptions (Gcap), (PI), (TJ), *the heat kernel lower bounds* (NLE) *is the best lower bound in some sense.*

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Remark

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Local case

Let us compare our *Main theorem* with the following result about the heat kernel of the Laplace-Beltrami operator on Riemannian manifolds. In this case consider the Poincaré inequality in the form

$$\int_B |f - f_B|^2 \, d\mu \leq Cr^2 \int_{2B} |\nabla f|^2 \, d\mu \tag{3}$$

for any $f \in C^1(2B)$. For example, (3) holds on any complete manifold of non-negative Ricci curvature.

Theorem

Let M be a geodesically complete Riemannian manifold and assume that M is α-regular (where necessarily $\alpha = \dim M$). Then the Poincaré inequality (3) is equivalent to the Gaussian estimate of the heat kernel

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/2}} \exp\left(-\frac{d^2(x,y)}{ct}\right).$$

1 Heat kernel and singular Dirichlet forms

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3 Main results
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4 Example

Definition

We say that *condition* ($J_{\geq}$) is satisfied if there exists a non-negative function J such that $dj(x,y) = J(x,y)d\mu(y)d\mu(x)$ in $M \times M$, and

$$J(x,y) \geq \frac{C}{V(x,d(x,y))\,W(x,d(x,y))}, \quad \mu \times \mu\text{-a.a. } x,y \in M,$$

where $C > 0$ is a constant independent of x, y.

Note that

$$(\mathrm{VD}) + (\mathrm{J}_{\geq}) \Rightarrow (\mathrm{PI}).$$

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Main results-2: Off-diagonal lower estimate

Theorem

Let $(\mathcal{E}, \mathcal{F})$ be a regular, pure jump type Dirichlet form in L^2. If conditions (VD), (RVD) *and* (TJ) *hold, and the jump kernel $J(x, y)$ exists, then*

$$(\mathrm{J}_{\geq}) + (\mathrm{Gcap}) \;\Leftrightarrow\; (\mathrm{LLE}) + (\mathrm{LE}).$$

Definition

We say that *condition* (LE) is satisfied if the heat kernel $p_t(x, y)$ exists, and there exists $C > 0$ such that for almost all $x, y \in M$ and $t > 0$,

$$p_t(x, y) \geq C \left(\frac{1}{V(x, W^{-1}(x, t))} \wedge \frac{t}{V(x, d(x, y))W(x, d(x, y))} \right).$$

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We say that condition (J) is satisfied if both conditions $(\mathrm{J}_{\geq})$ and $(\mathrm{J}_{\leq})$ are satisfied.

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Note that

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Two-sided heat kernel estimates

Corollary

Let $(\mathcal{E}, \mathcal{F})$ be a regular, pure jump type Dirichlet form in L^2. If conditions (VD), (RVD) *hold, then*

$$(\mathrm{Gcap}) + (\mathrm{J}) \Leftrightarrow (\mathrm{UE}) + (\mathrm{LE}).$$

Definition

We say that *condition* (UE) is satisfied if the heat kernel $p_t(x, y)$ exists, and there exists $C > 0$ such that for almost all $x, y \in M$ and $t > 0$,

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Lower estimates and Hölder regularity of the heat kernels for non-local Dirichlet form on doubling spaces

1. Heat kernel and singular Dirichlet forms
2. Stable-like type of heat kernel estimates
3. Main results
   - Near-diagonal lower estimates
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   - Main ingredients of proofs
4. Example

(i). Change of metric

In order to deal with the trouble arising from the dependence of $W(x,\cdot)$ on point x, we will introduce a new metric d_*. This idea is motivated by the following papers.

[1] M. T. Barlow and M. Murugan. Stability of elliptic Harnack inequality. *Ann. of Math. (2)*, 187(3):777–823, 2018.

[2] J. Kigami. Resistance forms, quasisymmetric maps and heat kernel estimates. *Mem. Amer. Math. Soc.*, 216(1015):vi+132, 2012.

By the arguments in [1], there exists a metric d_* on M which is equivalent to the old metric d.

Indeed, for any $r > 0$, let

$$B_*(x,r) := \{y \in M : d_*(y,x) < r\},$$

be a ball under the new metric d_*.

There exists $L_0 \geq 1$, such that for all x in M and all $r > 0$,

$$B_*(x, L_0^{-1} r) \subset B(x,R) \subset B_*(x,r),$$

where R is determined by r.

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For any $x \in M$ and any $r > 0$, let

$$V_*(x,r) := \mu(B_*(x,r)).$$

Proposition

Assume that (VD) *is satisfied. Then the following are true.*

1. *Condition* (VD$_*$) *holds true: there exists $C > 0$ such that, for all $x \in M$ and all $r > 0$,*

$$V_*(x,2r) \leq C V_*(x,r).$$

2. *Assume in addition that* (RVD) *is satisfied. Then, condition* (RVD$_*$) *holds true: there exists $\alpha'_* > 0$ such that for $x \in M$ and $0 < s \leq r$,*

$$\frac{V_*(x,r)}{V_*(x,s)} \geq C^{-1} \left(\frac{r}{s}\right)^{\alpha'_*}.$$

1. Under the new metric d_*, conditions (PI) and (TJ) can be rewritten in new shapes, denoted by (PI_*) and (TJ_*) respectively, where the function $W(x, r)$ becomes $W_*(r) := r^\beta$. Here, the universal number $\beta > 0$ is determined by the new metric d_*.

2. Condition (Gcap) can not be rewritten under the new metric d_*. However, in this case, condition (Gcap) can be replaced by a new and *equivalent* condition.

(ii). Weak Harnack inequality

Lemma (Weak Harnack inequality)

Assume that (VD$_*$), (RVD$_*$), (Gcap), (PI$_*$) *and* (TJ$_*$) *are satisfied. Then there exists* $\varepsilon \in (0,1)$ *depending only on the constants in the above hypotheses, such that the following is true: if a function* $u \in \mathcal{F}' \cap L^\infty$ *is superharmonic and non-negative in a ball* $2B_*$*, where* $B_* = B_*(x_0, r)$ *with* $r > 0$*, and if, for some* $a > 0$*,*

$$\frac{\mu((\kappa_* B_*) \cap \{u \geq a\})}{\mu(\kappa_* B_*)} \geq \frac{1}{2}$$

and

$$r^\beta \operatorname*{esup}_{z \in B_*} \int_{(2B_*)^c} u_-(y) J(z, dy) \leq \varepsilon a,$$

then

$$\operatorname*{einf}_{\frac{\kappa_*}{2} B_*} u \geq \varepsilon a.$$

Here κ_* is the constant as in (PI$_*$).

Remark

In local cases, weak Harnack inequality is equivalent to Harnack inequality, which implies the Hölder estimates of harmonic functions. For example,

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx,$$
$$\mathcal{F} := W^{1,2}(\mathbb{R}^n).$$

In our settings, we apply weak Harnack inequality to obtain the Hölder estimates of harmonic functions, and then the Hölder estimates of caloric functions. Consequently, we can obtain the Hölder estimates of Dirichlet heat kernels $p_t^{\Omega}(x,y)$ for bounded open sets $\Omega \subset M$.

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Lemma

Assume that (VD$_*$), (RVD$_*$), (Gcap), (PI$_*$) *and* (TJ$_*$) *are satisfied. Fix a ball* $B_*(x_0, R)$*. Then, for any open set* $\Omega \subset B_*(x_0, R)$*, the heat kernel* $p_t^\Omega(x, y)$ *exists, is jointly continuous in* $(x, y, t) \in \Omega \times \Omega \times (0, \infty)$ *and satisfies the upper bound*

$$\sup_{x,y\in\Omega} p_t^\Omega(x, y) \leq \frac{C}{V_*(x_0, R)} \left(\frac{R^\beta}{t}\right)^{\frac{1}{\nu}}.$$

and the following estimate: for any open subset $U \subset \Omega$ *and* $r > 0$ *with* $U_r \subset \Omega$*, we have for all* $x, x', y, y' \in U$ *and* $t > 0$*,*

$$\left|p_t^\Omega(x, y) - p_t^\Omega(x', y')\right| \leq \frac{C}{V_*(x_0, R)} \left(\frac{R^\beta}{t}\right)^{\frac{1}{\nu}} \left(\left(\frac{d_*(x, x')}{t^{1/\beta} \wedge r}\right)^\theta + \left(\frac{d_*(y, y')}{t^{1/\beta} \wedge r}\right)^\theta\right),$$

where $\theta \in (0, 1)$ *and* $C > 0$ *depend only on the constants in the hypothesis.*

Remark

The heat kernel $p_t(x, y)$ *also exists. However, it is not clear whether it is Hölder continuous or not under the hypothesis of the above lemma.*

Lemma

Assume that (VD$_*$), (RVD$_*$), (Gcap), (PI$_*$) *and* (TJ$_*$) *are satisfied. Fix a ball* $B_*(x_0, R)$. *Then, for any open set* $\Omega \subset B_*(x_0, R)$, *the heat kernel* $p_t^\Omega(x, y)$ *exists, is jointly continuous in* $(x, y, t) \in \Omega \times \Omega \times (0, \infty)$ *and satisfies the upper bound*

$$\sup_{x,y\in\Omega} p_t^\Omega(x, y) \le \frac{C}{V_*(x_0, R)} \left(\frac{R^\beta}{t}\right)^{\frac{1}{\nu}}.$$

and the following estimate: for any open subset $U \subset \Omega$ *and* $r > 0$ *with* $U_r \subset \Omega$, *we have for all* $x, x', y, y' \in U$ *and* $t > 0$,

$$\left|p_t^\Omega(x, y) - p_t^\Omega(x', y')\right| \le \frac{C}{V_*(x_0, R)} \left(\frac{R^\beta}{t}\right)^{\frac{1}{\nu}} \left(\left(\frac{d_*(x, x')}{t^{1/\beta} \wedge r}\right)^\theta + \left(\frac{d_*(y, y')}{t^{1/\beta} \wedge r}\right)^\theta\right),$$

where $\theta \in (0, 1)$ *and* $C > 0$ *depend only on the constants in the hypothesis.*

Remark

The heat kernel $p_t(x, y)$ *also exists. However, it is not clear whether it is Hölder continuous or not under the hypothesis of the above lemma.*

(iii). Reflected Dirichlet forms

To prove (LLE) $\Rightarrow$ (PI), we construct the reflected Dirichlet form on balls.
Fix a ball B and set

$$\overline{\mathcal{E}}(u,u) := \iint_{B\times B} (u(x)-u(y))^2 J(x,dy)\,d\mu(x), \quad u \in D[\overline{\mathcal{E}}],$$

$$D[\overline{\mathcal{E}}] := \{f|_B : f \in \mathcal{F}\}.$$

It follows from the theory in [1] and other arguments that $(\overline{\mathcal{E}}, D[\overline{\mathcal{E}}])$ is closable, and its closure $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ is a reflected Dirichlet from on $L^2(B,\mu)$.

[1] Z.-Q. Chen and M. Fukushima. *Symmetric Markov processes, time change, and boundary theory*, volume 35 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2012.

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Denote the semigroup of $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ by $\{\overline{P}_t\}$ and the semigroup of $(\mathcal{E}, \mathcal{F}(B))$ by $\{P_t^B\}$. It follows that

$$\overline{P}_t f \geq P_t^B f, \quad t > 0,\, f \geq 0,$$

which implies that for $u \in D[\overline{\mathcal{E}}] = \{f|_B : f \in \mathcal{F}\}$

$$\begin{aligned}
\overline{\mathcal{E}}(u,u) &\geq \frac{1}{t}(u - \overline{P}_t u, u) \\
&= \frac{1}{2t}(\overline{P}_t(u(x)1 - u)^2(x), 1(x)) \\
&\geq \frac{1}{2t}(P_t^B(u(x)1 - u)^2(x), 1(x)) \\
&= \frac{1}{2t}\int_B \int_B p_t^B(x,y)(u(x) - u(y))^2 d\mu(x) d\mu(y)
\end{aligned}$$

Then, applying (LLE) and taking the suitable t in the above inequality, one can obtain (PI).

1. Heat kernel and singular Dirichlet forms

2. Stable-like type of heat kernel estimates

3. Main results
   - Near-diagonal lower estimates
   - Off-diagonal lower estimates
   - Main ingredients of proofs

4. Example

Assume that $\beta \in (0, 2)$, $z_0 \in S^{d-1}$, $\theta \in [0, 1)$ and set

$$N := \{y \in \mathbb{R}^d : \langle \frac{y}{|y|}, z_0 \rangle \geq \theta\}.$$

On $\mathbb{R}^d$, consider the case that $dj(x, y) = J(x, y)dxdy$ and

$$\mathbf{1}_N(x - y)\frac{c}{|x - y|^{d+\beta}} \leq J(x, y) \leq \frac{C}{|x - y|^{d+\beta}}.$$

This example is shown in [1], where (PI) and regularity results are studied. While, the heat kernel is NOT studied.

Our result

$$(\text{Gcap}) + (\text{PI}) \Leftrightarrow (\text{LLE}) + (\text{NLE})$$

shows that the heat kernel $p_t(x, y)$ exists, is Hölder continuous jointly in (x, y), and satisfies (LLE) and (NLE).

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Thank you very much!