Elliptic Harnack inequalities and condition (UJS)

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The weak and strong elliptic Harnack inequalities

Basically, there are two different types of elliptic Harnack inequality. One is the weak Harnack inequality, and the other is the strong Harnack inequality. It turns out that they are equivalent to each other when the Dirichlet form is strongly local, where both the jump measure and the killing measure vanish.

However, the situation is quite different for the non-local DF, where the jump measure does not vanish. In fact, they cannot be equivalent in general. Of course, we always have

$$\text{strong} \Rightarrow \text{weak},$$

but the converse may not be true.

Main results: For any regular resurrected Dirichlet form, assume that (VD), (RVD) hold ▸ flowchart.

(1). Then

$$(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{wEH}) + (\mathrm{E}) + (\mathrm{J}_{\leq}).$$

(2). If further (UJS) holds, then

$$(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{sEH}) + (\mathrm{E}) + (\mathrm{J}_{\leq}).$$

(3). If (PI), (Gcap), $(\mathrm{J}_{\leq})$ are all satisfied, then

$$(\mathrm{UJS}) \Leftrightarrow (\mathrm{GJ}_{\geq})$$

(this equivalence is trivial when the DF is strongly local as $J = 0$).

(We will state what are these conditions.)

Our results: if condition (UJS) holds, then

$$\begin{aligned}(\mathrm{UE})+(\mathrm{LLE}) \quad &\Leftrightarrow \quad (\mathrm{wEH})+(\mathrm{E})+(\mathrm{J}_{\leq}) \\ &\Leftrightarrow \quad (\mathrm{sEH})+(\mathrm{E})+(\mathrm{J}_{\leq}).\end{aligned}$$

- If the DF is strongly local ($J \equiv 0$), the above equivalences become

$$\begin{aligned}(\mathrm{UE})+(\mathrm{LLE}) \quad &\Leftrightarrow \quad (\mathrm{wEH})+(\mathrm{E}) \\ &\Leftrightarrow \quad (\mathrm{sEH})+(\mathrm{E}).\end{aligned}$$

This conclusion was already obtained by Grigor'yan-Hu-Lau (2015, from Corollary 7.3 to Theorem 7.8). In fact, we obtained

weak Harnack $(\mathrm{wEH}) \Leftrightarrow (\mathrm{sEH})$ strong Harnack

for a strongly local DF when the measure satisfies the doubling condition, nothing else is required.

Classical elliptic Harnack inequality

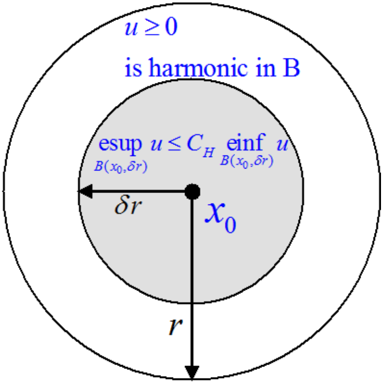
Our motivation

In 1961, Moser proved that, for any non-negative, **harmonic** function u in any ball B (with respect to the symmetric, uniformly elliptic divergence-form operator, which corresponds to a strongly local Dirichlet form in $L^2(dx)$)

$$\sup_{\frac{1}{2}B} u \leq C \inf_{\frac{1}{2}B} u. \tag{1}$$

The importance is that the constant $C \geq 1$ here is **universal** in the sense that it is independent not only of function u, but also of ball B. This inequality says that any non-negative harmonic function in a ball is **nearly constant** around the center. (There are **435** citations for this paper in Mathscinet.)

Nowadays, this inequality is called the elliptic Harnack inequality.



The oscillation of harmonic function around the center is small.

Since the celebrated work by Moser, there has been a lot of work devoted in this direction. Here we are only concerned with **equivalence** conditions for the heat kernel estimate involving **Harnack inequality**.

Heat kernel and Harnack inequality

In this direction,

- Grigor'yan-Telcs (2002) proved the following equivalence: for a random walk on an infinite, connected, locally finite graph (with a uniformly bounded number of edges)

$$(\mathrm{UE}_\beta) + (\mathrm{LE}_\beta) \Leftrightarrow (\mathrm{VD}) + (\mathrm{H}) + (\mathrm{E}).$$

condition (UE_β): $p_n(x,y) \le \frac{C}{V(x,n^{1/\beta})} \exp\left(-c\left(\frac{d(x,y)}{n^{1/\beta}}\right)^{\beta/(\beta-1)}\right)$.

condition (LE_β): for all $n \ge d(x,y) \vee 1$,

$p_n(x,y) + p_{n+1}(x,y) \ge \frac{C}{V(x,n^{1/\beta})} \exp\left(-c\left(\frac{d(x,y)}{n^{1/\beta}}\right)^{\beta/(\beta-1)}\right)$.

condition (E): $\mathbb{E}_x(\tau_{B(x,R)}) \asymp r^\beta$, where

$\tau_{B(x,R)} := \min\{n \ge 0 : X_n \notin B(x,R)\}$ is the **first exit time**.

Heat kernel and Harnack inequality

We generalize this result to the **metric space**.

- Grigor'yan-Hu (2014, Canadian JM) proved the following equivalence: for a strongly local Dirichlet form

$$(\mathrm{UE}_{\mathsf{exp}}) + (\mathrm{NLE}) \Leftrightarrow (\mathrm{H}) + (\mathrm{E})$$

if the **measure** is α-**regular**.

★ **condition** $(\mathrm{UE}_{\mathsf{exp}})$: $p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right)$.

★ **condition** (NLE): $p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}}$ whenever $d(x,y) \le \varepsilon t^{1/\beta}$.

★ **condition** (E): $\|G^B\mathbf{1}\|_{L^\infty} \le Cr^\beta$ and $\operatorname{einf}_{\delta B} G^B\mathbf{1} \ge C^{-1}r^\beta$ for any ball B of radius r, where $G^B\mathbf{1} = \int_0^\infty P_t 1\, dt$ is the **mean exit time** from ball B.

Question: What happens when the DF is **non-local** on a metric space?

This is the very issue we are going to address in this talk. Note that the above classical Harnack inequality

$$\sup_{\frac{1}{2}B} u \leq C \inf_{\frac{1}{2}B} u$$

no longer holds for non-local DF, since the **jump part** also plays a role.

- How does the **jump kernel** play a role in the **strong** Harnack inequality?

Weak elliptic Harnack inequality

The **weak elliptic Harnack** inequality:

$$\left(\fint_{B_r} u^p d\mu\right)^{1/p} \leq C\left(\operatorname*{einf}_{B_r} u + W(B_r) T_{\frac{3}{4}B_R,B_R}(u_-)\right)$$

for any two concentric balls $B_r \subset B_R$ and any non-negative harmonic function u in B_R, where

$$T_{\frac{3}{4}B_R,B_R}(u_-) := \operatorname*{esup}_{x\in\frac{3}{4}B_R} \int_{M\setminus B_R} u_-(y)J(x,y)d\mu(y)$$

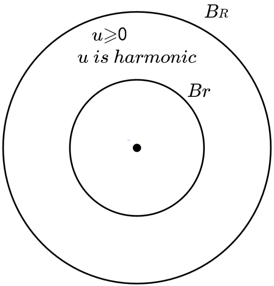
for a constant $0 < p < 1$ and

$$0 < R < \sigma\overline{R}, \quad 0 < r \leq \delta R,$$

with a localized parameter $\overline{R} \in (0,\infty]$ (possibly $\overline{R} = \infty$).

Weak elliptic Harnack inequality (continued)

The weak Harnack inequality says that the *average* of a powered function u^p of u for some $p \in (0,1)$ over a smaller ball B_r can be controlled by its *infimum* + the *product* of the scaling funciton with *a tail* of the negative part u_- of function u outside a bigger ball B_R.



$$\left(\frac{1}{\mu(B_r)} \int_{B_r} u^p \, d\mu \right)^{1/p} \leq C \left\{ \inf_{B_r} u + W(B_r) T_{B_{\frac{3}{4}R}, B_R}(u_-) \right\}$$

The weak elliptic Harnack inequality

$$\left(\fint_{B_r} u^p d\mu \right)^{1/p} \leq C \left(\operatorname*{einf}_{B_r} u + W(B_r) T_{\frac{3}{4}B_R, B_R}(u_-) \right). \tag{2}$$

◂ return seh

- If the Dirichlet form is *strongly local*, then the weak elliptic Harnack inequality becomes simpler

$$\left(\fint_{B_r} u^p d\mu \right)^{1/p} \leq C \operatorname*{einf}_{B_r} u \quad \text{(weak Harnack)}$$

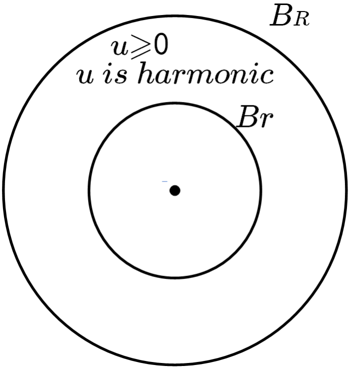
since $T_{\frac{3}{4}B_R, B_R}(u_-) \equiv 0$.

Strong elliptic Harnack inequality

The **strong elliptic Harnack** inequality

$$\operatorname*{esup}_{B_r} u \leq C \left(\operatorname*{einf}_{B_r} u + W(B_r) T_{\frac{3}{4}B_R, B_R}(u_-) \right), \tag{sEH}$$

that is, the left term in (2) ▸ wEH is replaced by the supremum.



$$\operatorname{esup}_{B_r} u \leq C \left\{ \operatorname*{einf}_{B_r} u + W(B_r) T_{B_{\frac{3}{4}R}, B_R}(u_-) \right\}$$

Clearly,

$$(\mathrm{sEH}) \Rightarrow (\mathrm{wEH}),$$

since it is clear that

$$\left(\fint_{B_r} u^p d\mu\right)^{1/p} \leq \operatorname*{esup}_{B_r} u.$$

The converse may **not** be true, even the basic **conditions** (PI), (TJ) **and** (E) (thus **condition** (Gcap) also holds), see Example 5.4 by **Chen-Kumagai-Wang(王健)** (2020, JEMS).

Framework (doubling space)

Our framework.

- (M, d, μ): a doubling space.

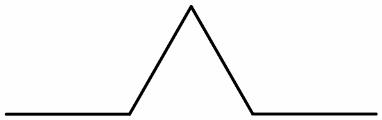
that is, (M, d) is a locally compact, separable metric space, and μ is a Radon measure (locally finite, inner regular) with full support ($\mu(\Omega) > 0$ for any open $\Omega \neq \emptyset$), which satisfies the doubling condition :

$$V(x, 2r) \leq CV(x, r) \text{ for all } x \text{ in } M \text{ and } r > 0,$$

where $V(x, r) := \mu(B(x, r))$ and $B(x, r)$ is an open metric ball.

The fractal is the desired model of the metric space in mind.

Von Koch curve (1904)

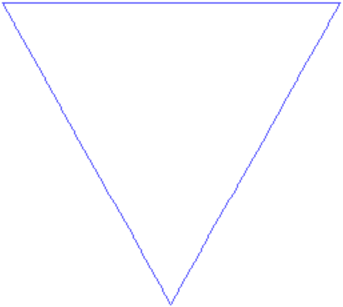


Start with a unit line, divide the line segment into three parts of equal length, replace the middle part with an equilateral triangle **but** removing the base, and for the four remaining segments, repeat the same procedure. Do this again and again, the limit is the Von-Koch curve.

Snowflake

Helge von Koch

(1870-1924), Sweden



Mathematical functions for Koch curve

Let

$$\begin{aligned} f_1(z) &= c\overline{z}, \\ f_2(z) &= (1-c)\overline{z} + c, \end{aligned}$$

where $c = \frac{1}{2} + \frac{1}{3}i$, and $\overline{z}$ is the conjugate of z. It turns out that the Koch curve K is uniquely determined by

$$K = f_1(K) \cup f_2(K).$$

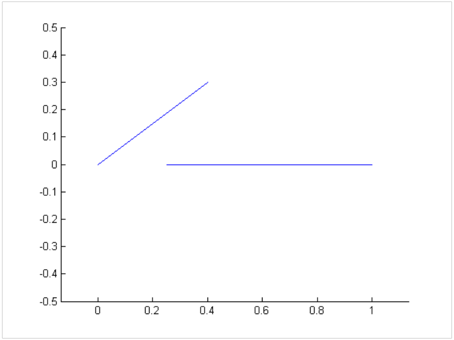
The Koch curve is called a **fractal**, or a **self-similar set**. The functions $\{f_1, f_2\}$ is called the **iterated function systems (IFS)**.

Hata's tree

If we keep f_1 the same expression but change the **function** f_2 to be

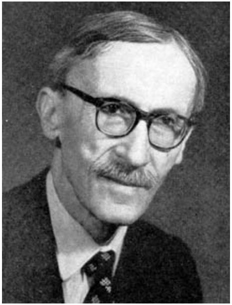
$$f_2(z) = (1 - |c|^2)\overline{z} + |c|^2,$$

and $c = 0.4 + 0.3i$. Imagine what happens?



Lévy curve, 1906

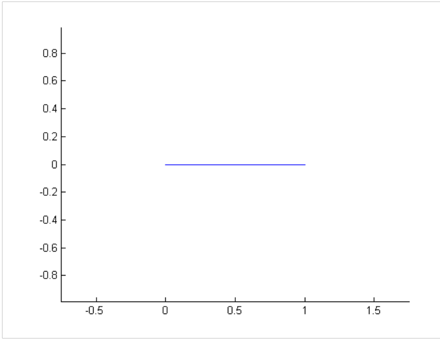
$f_1(z) = cz, \quad f_2(z) = (1-c)z + c$, where $c = 0.5 + 0.5i$.



Paul Pierre Lévy

(1886-1971),

France



The Lévy curve is also a **fractal**.

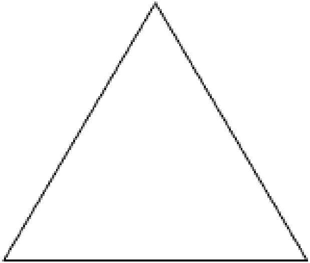
Sierpinski gasket, 1915

$$f_k(z) = \frac{1}{2}(z + p_k) \text{ for } k = 1, 2, 3.$$

W. F. Sierpiński

(1882-1969),

Poland



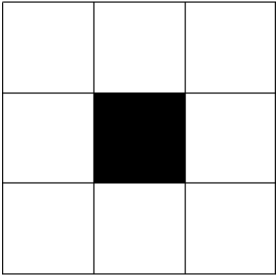
Sierpinski Carpet, 1916

$f_k(z) = \frac{1}{3}(z + 2p_k)$ for $1 \le k \le 8$.

W. F. Sierpiński

(1882-1969),

Poland



Fact

(Hutchinson, 1981). Let each f_i be a contraction on a complete metric space M. Then there exists a unique **non-empty compact** set $K \subset M$ such that

$$K = \bigcup_{i=1}^{N} f_i(K).$$

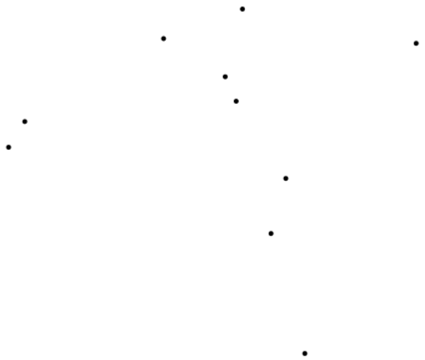
The set K called a self-similar set, or an invariant set, with respect to $\{f_i\}_{i=1}^N$ which is an iterated function system (IFS).

More examples: Fern 1

The **fern** can be generated by four simple functions.

More examples: tree 1

The **tree** can be generated by six simple functions.



More examples: tree 2

The **tree** can be generated by five simple functions.

Tree-3

The **following tree** is generated by the L-system.

n= 1

Framework (reverse volume doubling condition)

The measure μ also satisfies the following condition.

- The *reverse volume doubling condition* (RVD): There exist two positive constants $C \geq 1$ and d_1 such that for all $x \in M$ and $0 < r \leq R < \overline{R}$

$$\frac{V(x,R)}{V(x,r)} \geq C^{-1} \left(\frac{R}{r}\right)^{d_1}.$$

Known: if (M, d) is *connected* and *unbounded*, then

$$(VD) \Rightarrow (RVD),$$

see for example Grigor'yan-Hu (2014, Moscow JM).

Framework (the scaling function)

- The **scaling function** $W(x, r)$. The function $W(x, \cdot)$ is *continuous, strictly increasing*, $W(x, 0) = 0$, for any fixed x in M, and there exist positive constants C_1, C_2 and $\beta_2 \geq \beta_1$ such that for all $0 < r \leq R < \infty$ and all $x, y \in M$ with $d(x, y) \leq R$,

$$C_1 \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{W(x, R)}{W(y, r)} \leq C_2 \left(\frac{R}{r}\right)^{\beta_2}.$$

- For example, $W(x, r) = r^\beta$ for $\beta > 0$.

Heat kernel

The function $\{p_t\}_{t>0}$ (in three variables) is called a **heat kernel**, if

- symmetric: $p_t(x, y) = p_t(y, x)$;
- Markovian: $p_t(x, y) \geq 0$, and
$$\int_M p_t(x, y) d\mu(y) \leq 1;$$
- semigroup property:
$$p_{t+s}(x, y) = \int_M p_t(x, z) p_s(z, y) d\mu(z);$$
- identity approximation:
$$\|P_t f - f\|_2 \to 0 \text{ as } t \to 0$$

where $P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y)$.

Dirichlet form

- A **Dirichlet form** $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$ is a closed, Markovian, *symmetric form* (*that is bilinear, symmetric, non-negative definite with a densely defined domain*)
  - regular: $C_0(M) \cap \mathcal{F}$ is dense in both $\mathcal{F}$ and in $C_0(M)$.
  - local: $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with disjoint compact supports.
  - strongly local: $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with compact supports and f being constant in a neighborhood of $\mathrm{supp}(g)$.
- M. Fukushima and Y. Oshima and M. Takeda: **Dirichlet forms and symmetric Markov processes**, 2011, 2nd.

Example 1 (strongly local DF)

- The Gauss-Weierstrass function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

is a heat kernel (or a transition density of BM), which is the fundamental solution of the classical heat equation

$$\partial_t u - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n.$$

- The corresponding **Dirichlet form (local)** in L^2

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^n} \nabla f(x) \cdot \nabla g(x) dx = (-\Delta f, g).$$

Example 2 (non-local DF)

- The Cauchy-Poisson function

$$p_t(x,y) = \frac{C_n}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}}$$

with $C_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$ is also a heat kernel, which is a fundamental solution of the equation with fractional Laplacian

$$\partial_t u + (-\Delta)^{1/2} u = 0 \quad \text{in} \quad \mathbb{R}^n.$$

- The corresponding **Dirichlet form (non-local)** in L^2

$$\begin{aligned}\mathcal{E}(f,g) &= \frac{C_n}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+1}} dydx \\ &= \left((-\Delta)^{1/2} f, g\right).\end{aligned}$$

Framework (regular resurrected DF)

- $(\mathcal{E}, \mathcal{F})$: a **regular resurrected DF** in $L^2(M, \mu)$:

$$\mathcal{E}(u,u) = \mathcal{E}^{(L)}(u,u) + \iint_{M\times M} (u(x) - u(y))^2 dj(x,y).$$

The name '**resurrected**' was introduced by Fukushima-Oshima-Takeda on p. 186 in the book (2011).

- In this talk, note that the **jump kernel** J always exists so that

$$dj(x,y) = J(x,y)d\mu(x)d\mu(y).$$

(If $dj \equiv 0$ then $\mathcal{E} = \mathcal{E}^{(L)}$ is strongly local).

Condition (Gcap)

Generalized capacity condition (Gcap)**:**

- There exists a constant $C > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ (where $\mathcal{F}' = \mathcal{F} + \text{const}$) and for any two concentric balls B_R, B_{R+r},

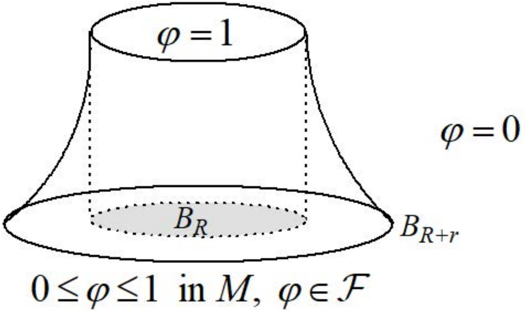
$$\mathcal{E}\left(u^2\varphi, \varphi\right) \leq C \sup_{x \in B_{R+r}} \frac{1}{W(x, r)} \int_{B_{R+r}} u^2 d\mu,$$

where $\varphi \in \text{cutoff}\left(B_R, B_{R+r}\right)$.

Note that here C is **universal** that is **independent** of balls B_R, B_{R+r} and functions u, φ.

The cutoff function

The cutoff function



The Poincaré inequality (PI):

- There exist two constants $\kappa \geq 1$, $C > 0$ such that for any metric ball $B := B(x_0, r)$ with $0 < r < \overline{R}/\kappa$ and any $u \in \mathcal{F}' \cap L^\infty$,

$$\int_B (u - u_B)^2 d\mu \leq Cw(B) \left\{ \int_{\kappa B} d\Gamma^{(L)}\langle u \rangle + \iint_{(\kappa B)\times(\kappa B)} (u(x) - u(y))^2 dj(x, y) \right\},$$

where u_B is the average of the function u over B

$$u_B = \frac{1}{\mu(B)} \int_B u d\mu =: \fint_B u d\mu$$

Stable-like upper estimate of heat kernel

- **Condition** (UE): if there exists a pointwise heat kernel $p_t(x,y)$ on $(0,\infty)\times M\times M$ such that, for any $x,y\in M$ and any $0<t<W(x,\overline{R})\wedge W(y,\overline{R})$,

$$p_t(x,y)\le C\left(\frac{1}{V(x,W^{-1}(x,t))}\wedge\frac{t}{V(x,y)W(x,y)}\right)$$

  with some positive constant C independent of t,x,y, where

$$\begin{aligned}V(x,y)&:=V(x,d(x,y)),\\ W(x,y)&:=W(x,d(x,y)).\end{aligned}$$

Stable-like upper estimate

- For example, if the metric space is unbounded with $\overline{R} = \infty$ and

$$V(x,r) = r^\alpha \quad \text{and} \quad W(x,r) = r^\beta \ \ (0 < \alpha, \beta < \infty),$$

then **condition** (UE) reads

$$p_t(x,y) \lesssim \frac{1}{t^{\alpha/\beta}} \wedge \frac{t}{d(x,y)^{\alpha+\beta}} \asymp \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}$$

for all $t > 0$ and all points x, y in M.

Localized lower estimate (LLE)

- **Condition** (LLE): if the following two properties are satisfied:
  1. for any bounded open set $\Omega \subset M$, the **Dirichlet heat kernel** $p_t^{\Omega}(x, y)$ **exists**;
  2. there exist $C > 0$ and $\varepsilon \in (0, 1)$ such that, for any ball $B := B(x_0, R)$ with $R \in (0, \overline{R})$ and any $0 < t \leq W(x_0, \varepsilon R)$,

$$p_t^B(x, y) \geq \frac{C^{-1}}{V(x_0, W^{-1}(x_0, t))}$$

for μ-almost all x, y in $B(x_0, \varepsilon w^{-1}(x_0, t))$.

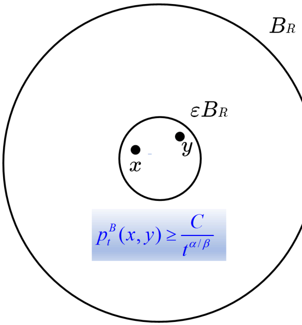
- For example, if

$$V(x,r) = r^{\alpha} \quad \text{and} \quad W(x,r) = r^{\beta} \ (0 < \alpha, \beta < \infty),$$

then **condition** (LLE) reads

$$p_t^B(x,y) \gtrsim \frac{1}{t^{\alpha/\beta}}$$

for μ-almost all points x, y in εB_R, whenever $0 < t < \varepsilon R^{\beta}$.



Mean exit time estimate

- **Condition** (E): if there exist two constants $C \geq 1$ and $\sigma \in (0,1)$ such that for any ball $B \subset M$ with radius less than $\sigma \overline{R}$,

$$\begin{aligned} \|G^B \mathbf{1}\|_{L^\infty} &\leq CW(B), \\ \operatorname*{einf}_{\delta B} G^B \mathbf{1} &\geq C^{-1} W(B) \end{aligned}$$

where $G^B \mathbf{1} := \int_0^\infty P_t^B 1 dt$, which satisfies that

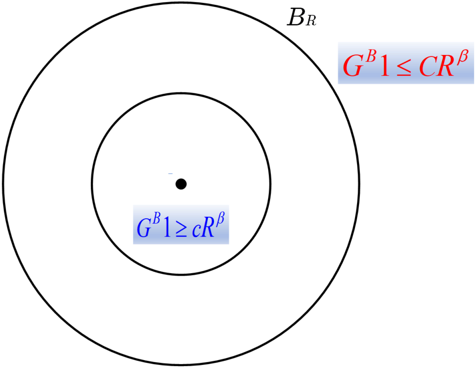
$$\mathcal{E}(G^B \mathbf{1}, \varphi) = \int_B \varphi \, \mathrm{d}\mu \text{ for any } \varphi \in \mathcal{F}(B).$$

Here $\{P_t^B\}_{t>0}$ is the **heat semigroup** associated with $(\mathcal{E}, \mathcal{F}(B))$.

- For example, if

$$W(x,r)=r^{\beta} \quad (0<\beta<\infty),$$

then **condition** (E) is indicated in the following picture



Upper bound of jump kernel

- **Condition** $(\mathrm{J}_{\leq})$: if the *jump kernel* $J(x, y)$ exists on $M \times M$, and there exists a positive constant C such that for $\mu \times \mu$-almost all points $(x, y) \in M \times M \setminus \mathrm{diag}$,

$$J(x, y) \leq \frac{C}{V(x, y)W(x, y)}.$$

Recall that

$$V(x, y) := V(x, d(x, y)) \quad \text{and} \quad W(x, y) := W(x, d(x, y)).$$

Theorem (Yu-H, 2023)

Let $(\mathcal{E}, \mathcal{F})$ be a regular resurrected Dirichlet form in L^2 and let $\overline{R} = \mathrm{diam}(M)$. If conditions (VD), (RVD) hold, then

$$(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{wEH}) + (\mathrm{E}) + (\mathrm{J}_{\leq}).$$

We use the fact that

- $(\mathrm{UE}) + (\mathrm{LLE}) \Leftrightarrow (\mathrm{PI}) + (\mathrm{Gcap}) + (\mathrm{J}_{\leq})$ by Grigor'yan-Hu-H (Theorem 2.20, 2023, to appear in Lau's volume).

The **key** argument

- $(\mathrm{PI}) + (\mathrm{Gcap}) + (\mathrm{TJ}) \Rightarrow (\mathrm{wEH})$ by H-Yu (Theorem 1.8, 2023, to appear in Lau's volume). A similar conclusion was obtained by Chen-Kumagai-Wang (2019) under the stronger assumptions.

Condition (UJS)

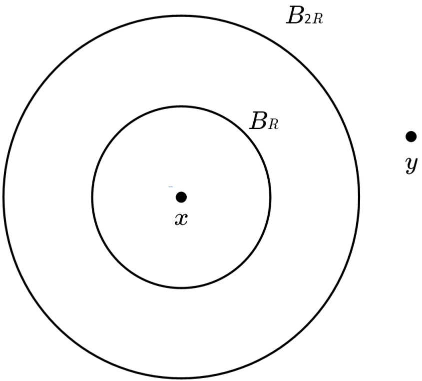
We further look at the equivalence condition for the heat kernel estimate involving the strong elliptic Harnack inequality. To do this, we need condition (UJS).

- **condition** (UJS) (meaning the *upper jumping smoothness* for the jump kernel): if the jump kernel $J(x, y)$ exists on $M \times M$ and satisfies that, for any $0 < R < \sigma\overline{R}$ and for $\mu \times \mu$-almost all points (x, y) in $M \times M$ with $d(x, y) \geq 2R$,

$$J(x, y) \leq C \fint_{B(x,R)} J(z, y) d\mu(z)$$

with some two positive constants C and $\sigma \in (0, 1)$ independent of x, y, R.

Condition (UJS) says that the value of the function $J(\cdot, y)$ at a point x is bounded from above by its average over a ball around this point, which reflects a certain degree of homogeneity of the function $J(\cdot, y)$.



$$J(x,y) \le C \frac{1}{\mu(B(x,R))} \int_{B(x,R)} J(z,y)\, d\mu(z)$$

This condition was introduced by **Barlow-Bass-Kumagai** (2009, Math Z.).

Examples

- $J \equiv 0$ (**strongly local DF**). Condition (UJS) is trivially satisfied.
- Condition (UJS) is satisfied for

  $$J(x,y) = \frac{1}{|x-y|^{n+\beta}}$$

  in $\mathbb{R}^n$, since for any point y with $|y-x| > 2R$ and any point z in $B(x,R)$,

  $$|z-y| \le |z-x| + |x-y| \le R + |x-y| \le \frac{3}{2}|x-y|,$$

  which gives that

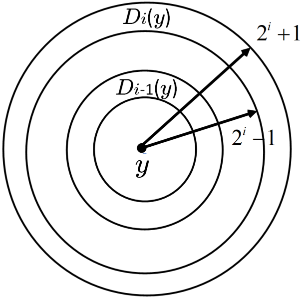
  $$\begin{aligned} \fint_{B(x,R)} J(z,y)dz &= \fint_{B(x,R)} \frac{1}{|z-y|^{n+\beta}} dz \\ &\ge \left(\frac{3}{2}|x-y|\right)^{-(n+\beta)} = \left(\frac{3}{2}\right)^{-(n+\beta)} J(x,y). \end{aligned}$$

- Condition (UJS) **fails**. Let $J(x, y)$ be defined in $\mathbb{R}^n$ by

$$J(x,y) := \frac{1}{|x-y|^{n+\beta}} + \frac{\mathbf{1}_D(x,y)}{|x-y|^{n+\theta}} \tag{3}$$

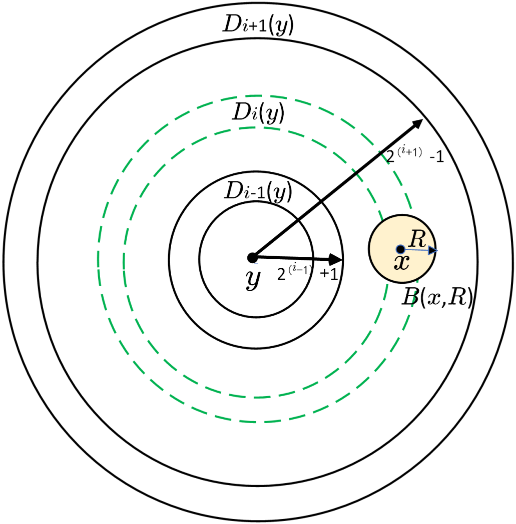
where $1 < \beta < 2$ and $1 \le \theta < \beta$, and the set $D \subset \mathbb{R}^n \times \mathbb{R}^n$ is the disjoint union of $\{D_i\}_{i=1}^{\infty}$: $D := \cup_{i=1}^{\infty} D_i$ with

$$D_i := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : 2^i - 1 < |x-y| < 2^i + 1\}.$$



$$D = \bigcup_{i=1}^{\infty} D_i$$

Condition (UJS) **fails** for any point (x, y) indiacted in the picture, since $J(x, y) \le C_J \fint_{B(x,R)} J(z, y)dz$ **fails** when $R \in [2^{i-2} - \frac{1}{2}, 2^{i-2} + \frac{1}{2}]$ for large i.



The gap between D_i and D_{i+1} is $2^i - 2$

However, in this example, the following basic conditions

$$(\text{TJ}), (\text{PI}), (\text{Gcap})$$

and condition

$$(\text{wEH})$$

are all satisfied for a regular Dirichlet form in $L^2(\mathbb{R}^n, dx)$ with the jump kernel $J(x, y)$ defined by (3), where the scaling function

$$W(x, r) = r^\beta.$$

Theorem (Hu-Yu, 2023)

Let $(\mathcal{E}, \mathcal{F})$ *be a regular resurrected Dirichlet form in* L^2 *and let* $\overline{R} = \mathrm{diam}(M)$*. If conditions* (VD), (RVD), (UJS) *are all satisfied, then*

$$\begin{aligned} (\mathrm{UE}) \ + (\mathrm{LLE}) \quad &\Leftrightarrow \quad (\mathrm{PI}) + (\mathrm{Gcap}) + (\mathrm{J}_{\le}) \\ &\Leftrightarrow \quad (\mathrm{wEH}) + (\mathrm{E}) + (\mathrm{J}_{\le}) \\ &\Leftrightarrow \quad (\mathrm{sEH}) + (\mathrm{E}) + (\mathrm{J}_{\le}). \end{aligned}$$

In particular, if the DF is strongly local, that is, $J \equiv 0$, then

$$\begin{aligned} (\mathrm{UE}) \ + (\mathrm{LLE}) \quad &\Leftrightarrow \quad (\mathrm{PI}) + (\mathrm{Gcap}) \\ &\Leftrightarrow \quad (\mathrm{wEH}) + (\mathrm{E}) \\ &\Leftrightarrow \quad (\mathrm{sEH}) + (\mathrm{E}), \end{aligned}$$

see Grigor'yan-Hu-Lau (2015).

The **key** argument: if (VD), (Gcap) and (FK) are all satisfied, then

$$(\mathrm{wEH}) + (\mathrm{J}_{\leq}) + (\mathrm{UJS}) \Rightarrow (\mathrm{sEH}) \quad \text{(H-Yu, 2023)}.$$

We **remark** that Chen-Kumagai-Wang (Theorem 1.11 (i), 2019) proved that if (VD), (CSJ) and (FK) are all satisfied, then

$$(\mathrm{wEH}) + (\mathrm{J}_{\leq}) + (\mathrm{J}_{\geq}) \Rightarrow (\mathrm{sEH}).$$

Our result here is sharper in the sense that

$$(\mathrm{J}_{\leq}) + (\mathrm{J}_{\geq}) \Rightarrow (\mathrm{UJS}),$$

and there is an example that condition (UJS) is true but $(\mathrm{J}_{\geq})$ fails.

Consequently, if conditions (VD) and (RVD) are satisfied, then

$$(\mathrm{Gcap}) + (\mathrm{PI}) + (\mathrm{J}_{\leq}) + (\mathrm{UJS}) \Rightarrow (\mathrm{sEH}) \quad \text{(H-Yu, 2023).} \tag{4}$$

Note that condition (UJS) is used to show the so-called condition (TE), as we will see below.

Assertion (H.-Yu,2023). Assume that $(\mathcal{E}, \mathcal{F})$ is a regular resurrected Dirichlet form in L^2. If conditions (VD), $(\mathrm{Cap}_{\leq})$ are satisfied, then

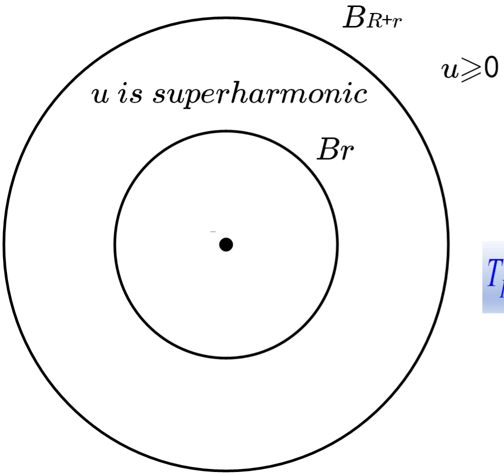
$$(\mathrm{TJ}) + (\mathrm{UJS}) \Rightarrow (\mathrm{TE}). \tag{5}$$

This is the **very place** we have used (UJS)!

Condition (TE): there exist three positive constants C, c_1 and $\sigma \in (0, 1)$ such that, for any two concentric balls $B_R = B(x_0, R)$, $B_{R+r} = B(x_0, R + r)$ with $0 < R < R + r < \sigma\overline{R}$, for any **globally non-negative** function $u \in \mathcal{F}' \cap L^\infty$ that is superharmonic in B_{R+r} with $\operatorname{esup}_{B_{R+r}} u > 0$, we have

$$T_{B_R, B_{R+r}}(u) = \operatorname*{esup}_{x \in B_R} \int_{B^c_{R+r}} u(y) J(x, dy) \leq \frac{C}{W(x_0, r)} \left(\frac{R + r}{r}\right)^{c_1} \operatorname*{esup}_{B_{R+r}} u.$$

The importance of this inequality lies in the property that the **supremum** of a superharmonic function u over the *ball* B_{R+r} (instead of over its complement B^c_{R+r}) can control the tail $T_{B_R,B_{R+r}}(u)$.



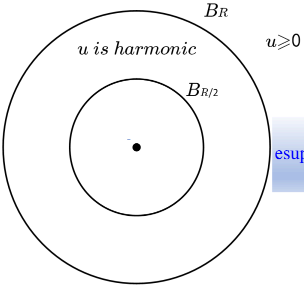
$T_{B_R,B_{R+r}}(u)$ is bounded by $\operatorname{esup}_{B_{R+r}} u$

Using (TE), together with (FK), (Gcap), (TJ), we have the following **mean-value inequality (MV):**

$$\operatorname*{esup}_{\frac{1}{2}B_R} u \le C\left[\delta^{-\theta}\left(\fint_{B_R} u^2 d\mu\right)^{1/2} + \delta \operatorname*{esup}_{B_R} u\right]$$

for any $\delta \in (0, 1]$ and any **globally non-negative** function $u \in \mathcal{F}' \cap L^\infty$ that is *harmonic* in any ball $B_R := B(x_0, R)$ with $0 < R < \sigma\overline{R}$.

We emphasize that the constants C, θ, σ are independent of number δ, ball B_R, function u.



$$\operatorname{esup}_{\frac{1}{2}B_R} u \le C\left\{\delta^{-\theta}\left(\frac{1}{\mu(B_R)}\int_{B_R} u^2 d\mu\right)^{1/2} + \delta \operatorname{esup}_{B_R} u\right\}$$

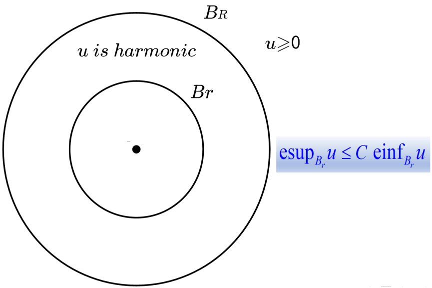
After that, we need to show the following implications

$$(\mathrm{MV}) + (\mathrm{wEH}) \Rightarrow (\mathrm{EHI}),$$

$$(\mathrm{FK}) + (\mathrm{J}_{\leq}) + (\mathrm{EHI}) \Rightarrow (\mathrm{sEH}),$$

which finishes the proof.■

- **condition** (EHI): for any **globally non-negative** function $u \in \mathcal{F}' \cap L^\infty$ that is harmonic in B_R, we have $\operatorname{esup}_{B_r} u \leq C \operatorname{einf}_{B_r} u$ for any $0 < r \leq \delta R < R < \sigma \overline{R}$.



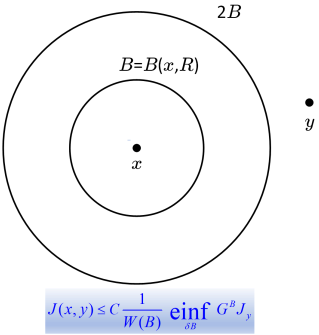
We are looking for an **equivalent** condition for (UJS). For this, we introduce $(\mathrm{GJ}_{\geq})$.

- We say that **condition** $(\mathrm{GJ}_{\geq})$ holds if there exist three constants $C > 0$ and δ, σ in $(0, 1)$ such that, for any $0 < R < \sigma \overline{R}$ and for $\mu \times \mu$-almost all points (x, y) in $M \times M$ with $d(x, y) \geq 2R$,

$$J(x, y) \leq \frac{C}{W(B)} \operatorname*{einf}_{\delta B} G^B J_y, \tag{6}$$

where $B := B(x, R)$ and $J_z(x) := J(x, z)$. Here for an open set Ω, the G^{Ω} is the *Green operator* G^{Ω}, which is the inverse of the generator of the form $(\mathcal{E}, \mathcal{F}(\Omega))$.

Condition $(\mathrm{GJ}_{\geq})$



One may compare two inequalities:

$$J(x,y) \lesssim \fint_B J(\cdot, y)\,d\mu \quad \text{condition (UJS)}$$

$$J(x,y) \lesssim \frac{1}{W(B)} \operatorname{einf}_{\delta B} G^B J_y \quad \text{condition } (\mathrm{GJ}_{\geq}).$$

Proposition. Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2. If condition (VD) holds, then

$$(\mathrm{E}_{\le}) + (\mathrm{GJ}_{\ge}) \Rightarrow (\mathrm{UJS}). \tag{7}$$

Proof. Let $0 < R < \sigma\overline{R}$ and $B := B(x,R)$, $d(x,y) \ge 2R$.

Step 1. By $(\mathrm{E}_{\le})$, we know that

$$\begin{aligned}
\operatorname*{einf}_{z\in\delta B} G^B J_y(z) &\le \frac{1}{\mu(\delta B)}\int_{\delta B} G^B J_y(z)d\mu(z) \\
&\le \frac{1}{\mu(\delta B)}\|G^B J_y\|_{L^1(B)} \\
&\le \frac{1}{\mu(\delta B)}\|G^B \mathbf{1}_B\|_{L^\infty}\|J_y\|_{L^1(B)} \\
&\lesssim \frac{W(B)}{\mu(\delta B)}\|J_y\|_{L^1(B)} \quad (\text{using } (\mathrm{E}_{\le})) \\
&\lesssim W(B) \fint_B J(z,y)d\mu(z).
\end{aligned}$$

Step 2. By $(\mathrm{GJ}_{\geq})$, we know that

$$\begin{aligned} J(x,y) &\lesssim \frac{1}{W(B)} \operatorname*{einf}_{\delta B} G^B J_y \\ &\lesssim \frac{1}{W(B)} \left\{ W(B) \fint_B J(z,y) d\mu(z) \right\} \\ &\lesssim \fint_B J(z,y) d\mu(z), \end{aligned}$$

thus showing condition (UJS). ■

The opposite implication from condition (UJS) to condition $(\mathrm{GJ}_{\geq})$ is more complicated and is needed more assumptions, as we will see below.

Theorem (H.-Yu, 2023)

Let $(\mathcal{E}, \mathcal{F})$ *be a regular resurrected Dirichlet form in* L^2*. If the basic conditions* (VD), (RVD), (PI), (Gcap)*, as well as condition* $(\mathrm{J}_{\leq})$*, are all satisfied, then*

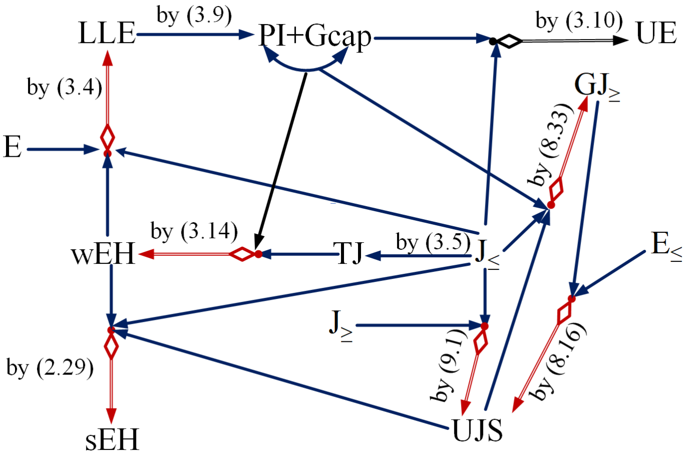
$$(\mathrm{UJS}) \Leftrightarrow (\mathrm{GJ}_{\geq}).$$

The key steps:

Step 1. $(\mathrm{E}_{\leq}) + (\mathrm{GJ}_{\geq}) \Rightarrow (\mathrm{UJS})$ by using (7) (easy!).

Step 2. $(\mathrm{Gcap}) + (\mathrm{PI}) + (\mathrm{J}_{\leq}) + (\mathrm{UJS}) \Rightarrow (\mathrm{GJ}_{\geq})$ (hard!).

In **summary**, the key implications in this talk are indicated in **red** color ◂ return.



The end of talk.