Heat kernel and Green function on subgraphs of a complete graph

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1. Definitions and notions

Let $K = K_N$ be a complete graph with N vertices. Let

$$V_0 := \{1, 2, \ldots, N\},$$
$$V_1 := \{ij : i, j \in V_0, i \neq j\},$$
$$V_2 := \{ijk : i, j, k \in V_0, i \neq j \text{ and } j \neq k\}$$

denote, respectively, the set of vertices, directed edges, and directed paths of length two.
For $n \geq 1$, let

$$V_n := \{i_0 \cdots i_n : i_0, \ldots, i_n \in V_0, i_j \neq i_{j+1} \text{ for all } j = 0, \ldots, n-1\}$$

denote the set of directed paths of length n.

Let G be a subgraph of K with vertex set $V_0^G \subseteq V_0$ and edge set $V_1^G \subseteq V_1$. For $n \geq 1$, let

$$V_n^G := \{i_0 \cdots i_n \in V_n : i_j i_{j+1} \in V_1^G \text{ for all } j = 0, \ldots, n-1\}$$

denote the set of directed paths in the graph G.
We let G^c be the complement of G defined as follows. Let $V_0^{G^c} := V_0 \setminus V_0^G$ and call it the set of vertices of G^c. Let $V_1^{G^c} := V_1 \setminus V_1^G$.

For each $n \geq 1$, let

$$V_n^{G^c} := V_n \setminus V_n^G \tag{1}$$

be the set of directed paths of length n associated with G^c. Note that a directed path in $V_n^{G^c}$ may contain a subpath that belong to some V_k^G, $1 \leq k \leq n-1$.

For each $n \geq 0$, we call any real-valued function on V_n an *n-form* on V_n, and let Λ^n be the vector space of all n-forms on V_n. Let $\{e^{i_0 \cdots i_n}\}_{i_0 \cdots i_n \in V_n}$ be the canonical basis on Λ^n with $e^{i_0 \cdots i_n}$ taking the value 1 at $i_0 \cdots i_n$ and zero elsewhere.

Define the *exterior operator* $d_n = d_n^K : \Lambda^n \to \Lambda^{n+1}$ as follows. For

$$\omega = \sum_{i_0 \cdots i_n \in V_n} \omega_{i_0 \cdots i_n} e^{i_0 \cdots i_n} \in \Lambda^n, \tag{2}$$

define

$$(d_n \omega)_{i_0 \cdots i_{n+1}} := \sum_{k=0}^{n+1} (-1)^k \omega_{i_0 \cdots \hat{i}_k \cdots i_{n+1}},$$

where $\hat{i}_k$ means that the index i_k is removed. For each $n \geq 0$, we also define d_n^G and $d_n^{G^c}$ as follows. Let ω be as in (2).
Then

$$d_n^G(\omega) := \sum_{i_0 \cdots i_n \in V_n} \omega_{i_0 \cdots i_n} d_n^G(e^{i_0 \cdots i_n}),$$

where

$$(d_n^G(e^{i_0 \cdots i_n}))_{j_0 \cdots j_{n+1}} := \begin{cases} (d_n(e^{i_0 \cdots i_n}))_{j_0 \cdots j_{n+1}} & \text{if } j_0 \cdots j_{n+1} \in V_{n+1}^G, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Define

$$d_n^{G^c}(\omega) := \sum_{i_0\cdots i_n \in V_n} \omega_{i_0\cdots i_n} d_n^{G^c}(e^{i_0\cdots i_n}),$$

where

$$(d_n^{G^c}(e^{i_0\cdots i_n}))_{j_0\cdots j_{n+1}} := \begin{cases} (d_n(e^{i_0\cdots i_n}))_{j_0\cdots j_{n+1}} & \text{if } j_0\cdots j_{n+1} \in V_{n+1}^{G^c}, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

It follows directly from the above definitions that

$$d_n = d_n^G + d_n^{G^c}. \tag{5}$$

Example 1 Consider the complete graph K_3 with vertices $\{1, 2, 3\}$. Let G be the complete subgraph with the vertices $\{1, 2\}$. Then

$$V_0^G = \{1, 2\}, \quad V_1^G = \{12, 21\}, \quad V_2^G = \{121, 212\},$$
$$V_0^{G^c} = \{3\}, \quad V_1^{G^c} = \{13, 23, 31, 32\},$$
$$V_2^{G^c} = V_2^K \setminus V_2^G = \{123, 131, 132, 213, 231, 232, 312, 313, 321, 323\}.$$

$$d_0^G = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad d_0^{G^c} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Notice that $d_0 = d_0^G + d_0^{G^c}$.

We denote 1-form on K_N as ω_{ij}. Suppose $N > 3$, then there are following 6 different heat kernels of 1-forms on K_N:

(1) $H_t(\omega_{ij}, \omega_{ij}) := u_1(t)$;
(2) $H_t(\omega_{ij}, \omega_{ji}) := u_2(t)$;
(3) $H_t(\omega_{ij}, \omega_{jk}) := u_3(t)$;
(4) $H_t(\omega_{ij}, \omega_{kj}) := u_4(t)$;
(5) $H_t(\omega_{ij}, \omega_{ik}) := u_5(t)$;
(6) $H_t(\omega_{ij}, \omega_{kl}) := u_6(t)$,

where index on same function are all different.

And the heat kernel satisfies the heat equation for 1-form: where $\Delta H_t(\omega_{ij},\cdot) = \sum_{\alpha\neq i,j}[3H_t(\omega_{ij},\cdot) - H_t(\omega_{\alpha j},\cdot) - H_t(\omega_{i\alpha},\cdot)]$, $\frac{\partial}{\partial t}H_t(\omega_{ij},\cdot) = \Delta H_t(\omega_{ij},\cdot)$. Then we obtain the 6 ODE with initial values

$$H_0(\omega_{ij},\omega_{kl}) = \begin{cases} 1 & \text{if } \omega_{kl} = \omega_{ij} \\ 0 & \text{oterwise.} \end{cases}$$

By solving a system of ODEs we get

$$
\begin{aligned}
u_1(t) =& \frac{1}{N(N-1)} \cdot \Big(e^{-(N+2)t} + (N-1)e^{-2(N+1)t} + (N-1)e^{-2Nt} \\
& + (N^2 - 3N + 1)e^{-3Nt} \Big), \\
u_2(t) =& \frac{1}{N(N-1)} \Big(e^{-(N+2)t} + (N-1)e^{-2(N+1)t} - (N-1)e^{-2Nt} - e^{-3Nt} \Big), \\
u_3(t) =& \frac{1}{2N(N-1)(N-2)} \cdot \Big(2(N-2)e^{-(N+2)t} + (N-1)(N-4)e^{-2(N+1)t} \\
& - (N-1)(N-2)e^{-2Nt} + 2e^{-3Nt} \Big), \\
u_4(t) =& \frac{1}{2N(N-1)(N-2)} \cdot \Big(2(N-2)e^{-(N+2)t} + (N-1)(N-4)e^{-2(N+1)t} \\
& + (N-1)(N-2)e^{-2Nt} - 2(N^2 - 3N + 1)e^{-3Nt} \Big), \\
u_5(t) =& \frac{1}{2N(N-1)(N-2)} \cdot \Big(2(N-2)e^{-(N+2)t} + (N-1)(N-4)e^{-2(N+1)t} \\
& + (N-1)(N-2)e^{-2Nt} - 2(N^2 - 3N + 1)e^{-3Nt} \Big), \\
u_6(t) =& \frac{1}{N(N-1)(N-2)} \cdot \Big((N-2)e^{-(N+2)t} - 2(N-1)e^{-2(N+1)t} + Ne^{-3Nt} \Big).
\end{aligned}
$$

We observe that $u_i(t)$, $i = 1, \ldots, 6$, can be re-written as

$$u_i(t) = \frac{e^{-(N+2)t}}{N(N-1)}\big(1 + \widetilde{u}_i(t)\big), \qquad i = 1, \ldots, 6, \tag{6}$$

where $\widetilde{u}_i(t)$ is bounded on $[0, \infty)$ and $\widetilde{u}_i(t) \to 0$ as $t \to \infty$.

2. Recursive formula for heat kernels on n-forms of a subgraph

Let $\Delta_0 := (d_0^K)^* d_0^K$ be the Laplacian on 0-forms, where A^* denotes the transpose of a A. For $n \geq 1$, let

$$\Delta_n^K := (d_n^K)^* d_n^K + d_{n-1}^K (d_{n-1}^K)^*$$

be the Laplacian on n-forms. Define Δ_n^G and $\Delta_n^{G^c}$ analogously.

Proposition 2 The following relations hold.

(a) For any $n \geq 0$, $(d_n^G)^*(d_n^{G^c}) = 0$ and $(d_n^{G^c})^*(d_n^G) = 0$.

(b) $\Delta_0^K = \Delta_0^G + \Delta_0^{G^c}$.

(c) For any $n \geq 1$,
$\Delta_n^K = \Delta_n^G + \Delta_n^{G^c} + d_{n-1}^G (d_{n-1}^{G^c})^* + d_{n-1}^{G^c} (d_{n-1}^G)^*$.

To simplify notation we let

$$L_{-1} := 0 \qquad \text{and} \qquad L_{n-1} := d^G_{n-1}(d^{G^c}_{n-1})^* + d^{G^c}_{n-1}(d^G_{n-1})^* \quad \text{for } n \geq 1.$$

Proposition 3 Let $n \geq 0$. Then for all $x, y \in V_n$ and $t \geq s \geq 0$,

$$H^G_t(x,y) = H^K_t(x,y) - \int_0^t \big(H^K_{t-s}(\Delta^{G^c}_n + L_{n-1})H^G_s\big)(x,y)\,ds.$$

Let $\mathcal{F}$ be the vector space of all real-valued functions on $[0,\infty)\times V_n^2$. Let $T:\mathcal{F}\to\mathcal{F}$ be a linear operator defined as

$$Tf_t(x,y) := -\int_0^t \big(H^K_{t-s}(\Delta_n^{G^c}+L_{n-1})\big) f_s(x,y)\,ds. \tag{7}$$

Proposition 4 Let T be defined as in (7) and assume $\| T \| < 1$. Then

$$H^G_t(x,y) = (I+T+T^2+\cdots)H^K_t(x,y).$$

3. Heat kernel on 0-forms

A complete graph K_N has N vertices and $K(N-1)/2$ edges. The combinatorial Laplacian has eigenvalues 0 (with multiplicity 1) and N (with multiplicity $N-1$). The normalized Laplacian has eigenvalues 0 (with multiplicity 1) and $N/(N-1)$ (with multiplicity $N-1$). Let V be the set of vertices of K_N. Let G be a sub-graph of K_N with N vertices. Let G^c denote the complement of G obtained by removing those edges in K_N that appear in G.
Recall that the combinatorial Laplacian Δ on a graph is defined as $\Delta = A - D$, where A and D are the adjacency and degree matrices respectively. Let $H_t^K(x,y), H_t^G(x,y), H_t^{G^c}(x,y)$ denote the combinatorial Laplacians corresponding to K, G, G^c respectively.

We use similar notation for the Laplacian Δ and the degree d_x of an element. Then

$$\Delta^K = \Delta^G + \Delta^{G^c}.$$

Proposition 5 For all $x, y \in V$ and $t \geq 0$,

$$H_t^G(x,y) = H_t^K(x,y) - e^{-Nt} \int_0^t e^{Ns} \Delta_x^{G^c} H_s^G(x,y)\, ds.$$

Now let $\mathcal{F}$ be the space of all real-valued functions on $[0, \infty) \times V$. Let $T : \mathcal{F} \to \mathcal{F}$ be a linear operator defined as

$$Tu(t, x) := -e^{-Nt} \int_0^t e^{Ns} \Delta_x^{G^c} u(s, x)\, ds.$$

Proposition 6 If $t < \log 2/N$, then $\|T\| < 1$.

Under the hypothesis of Proposition 6, $\|T\| < 1$ and thus by using Proposition 5,

$$H_t^G(x, y) = (I + T + T^2 + \cdots) H_t^K(x, y). \tag{8}$$

To derive a more explicit formula for $H_t^G(x,y)$, for each $y \in V$, we let $u_y : V \to V$ be the function defined as

$$u_y^{G^c}(x) := \begin{cases} d_x^{G^c}, & \text{if } x = y, \\ 0, & \text{if } x \neq y \text{ and } x \underset{G}{\sim} y, \\ -1, & \text{if } x \neq y \text{ and } x \underset{G^c}{\sim} y, \end{cases} \tag{9}$$

where $d_x^{G^c}$ is the number of neighbors of x in G^c.

Proposition 7 Let $u_y : V \to V$ be defined as in (9). Then for any $x, y \in V$, and all $t \geq 0$,

$$H_t^G(x,y) = H_t^K(x,y) + te^{-Nt}u_y^{G^c}(x) + e^{-Nt}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}t^m}{m!}\left(\Delta_x^{G^c}\right)^{m-1}u_y^{G^c}(x)$$

$$= \begin{cases} \frac{1}{N} - \frac{1}{N}e^{-Nt} + e^{-Nt}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m!}\left(\Delta_x^{G^c}\right)^{m-1}u_y^{G^c}(x), & y \neq x, \\ \\ \frac{1}{N} + (1-\frac{1}{N})e^{-Nt} + e^{-Nt}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m!}\left(\Delta_x^{G^c}\right)^{m-1}u_y^{G^c}(x), & y = x, \end{cases}$$

where $\left(\Delta_x^{G^c}\right)^{m-1}$ denotes the $(m-1)$-fold composition of $\Delta_x^{G^c}$ (or equivalently, the $(m-1)$th power of $\Delta_x^{G^c}$).

Corollary 8 The formula in Proposition 7 can be simplified as

$$H_t^G(x,y) = \begin{cases} H_t^K(x,y)\Big(1 + \frac{Nt}{e^{Nt}-1} u_y^{G^c}(x) + \sum_{m=2}^{\infty} \frac{Nt^m}{m!(e^{Nt}-1)} \big(\Delta_x^{G^c}\big)^{m-1} u_y^{G^c}\big)(x)\Big), & y \neq x, \\ H_t^K(x,y)\Big(1 + \frac{Nt}{e^{Nt}+N-1} u_y^{G^c}(x) + \sum_{m=2}^{\infty} \frac{Nt^m}{m!(e^{Nt}+N-1)} \big(\Delta_x^{G^c}\big)^{m-1} u_y^{G^c}\big)(x)\Big), & y = x. \end{cases}$$

Proposition 9 Consider the expansion in Corollary 8, the radius of convergence of each of the following series

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} t^m}{m!} \left(\Delta_x^{G^c}\right)^{m-1} u_y^{G^c}(x)$$

and

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} t^{m-1}}{m!} \left(\Delta_x^{G^c}\right)^{m-1} u_y^{G^c}(x)$$

is ∞.

4. Computing the coefficients in the heat kernel expansion

Let $c_k(x,y)$ be the coefficients of t^k in the expansion of $H_t^G(x,y)$, that is,

$$H_t^G(x,y) = H_t^K(x,y)\big(c_0(x,y) + c_1(x,y)t + c_2(x,y)t^2 + \cdots\big).$$

Then from the above results we get

$$c_0(x,y) := \begin{cases} 1, & \text{if } x = y, \\ 1, & \text{if } x \neq y \text{ and } x \underset{G}{\sim} y, \\ 0, & \text{if } x \neq y \text{ and } x \underset{G^c}{\sim} y. \end{cases}$$

To compute the other coefficients, we let $\eta^{G,G}(x,y)$ be the number of G-neighbors of x that are also G-neighbors of y, and let $\eta^{G,G^c}(x,y)$ be the number of G-neighbors of x that are G^c-neighbors of y. Similarly, we define $\eta^{G^c,G}(x,y)$ and $\eta^{G^c,G^c}(x,y)$. Since K is a complete graph, we have

$$d_x^G + d_x^{G^c} = N - 1.$$

Notice that $\eta^{G,G}(w,x)$ equals the number of triangles with one side being the edge connecting w and x, and the other two sides being edges in G. Hence $\sum_{w \underset{G^c}{\sim} x} \eta^{G,G}(w,x)$ is the total number of triangles with one vertex at x, one side being an edge in G^c connecting x, and the other two sides being edges in G. Thus we let

$$\eta_{\blacktriangle}^{G^c;G,G}(x) = \sum_{w \underset{G^c}{\sim} x} \eta^{G,G}(w,x).$$

Proposition 10 For any $x, y \in V_0$,

$$c_1(x,y) = \begin{cases} d_x^{G^c}, & \text{if } x = y, \\ \frac{1}{2}\eta^{G^c,G^c}(x,y)\big), & \text{if } x \neq y \text{ and } x \underset{G}{\sim} y, \\ \frac{1}{2}\big(N - d_x^{G^c} - d_y^{G^c} + \eta^{G^c,G^c}(x,y)\big), & \text{if } x \neq y \text{ and } x \underset{G^c}{\sim} y, \end{cases}$$

$$= \begin{cases} N - 1 - d_x^{G}, & \text{if } x = y, \\ \frac{1}{2}\big(N - d_x^{G} - d_y^{G} + \eta^{G,G}(x,y)\big), & \text{if } x \neq y \text{ and } x \underset{G}{\sim} y, \\ \frac{1}{2}\eta^{G,G}(x,y), & \text{if } x \neq y \text{ and } x \underset{G^c}{\sim} y. \end{cases}$$

Proposition 11 For any $x, y \in V_0$, the following hold.

(a) If $x = y$, we have

$$c_2(x,x) = \frac{1}{2}\big(N^2 - 3N + 2 + (3-2N)d_x^G + (d_x^G)^2\big).$$

(b) If $x \neq y$ and $x \underset{G}{\sim} y$, then

$$\begin{aligned} c_2(x,y) =& \frac{1}{12}\big(N^2 + (2-3N)d_x^G - 3Nd_y^G + 2(d_x^G)^2 + 2(d_y^G)^2 \\ &+ 2d_x^G d_y^G - (2 - 3N + 2d_x^G + 2d_y^G)\eta^{G,G}(x,y) \\ &- 2\sum_{w \underset{G^c}{\sim} x, w\neq y} \eta^{G,G}(w,y) + 2\sum_{w \underset{G^c}{\sim} x, w \neq y, w \underset{G}{\sim} y} d_w^G\big). \end{aligned}$$

(c) If $x \neq y$ and $x \underset{G^c}{\sim} y$, then

$$\begin{aligned} c_2(x,y) =& \frac{1}{12}\big(-2d_y^G + (3N - 2 - 2d_x^G - 2d_y^G)\eta^{G,G}(x,y) \\ &- 2\sum_{w \underset{G^c}{\sim} x, w\neq y} \eta^{G,G}(w,y) + 2\sum_{w \underset{G^c}{\sim} x, w \neq y, w \underset{G}{\sim} y} d_w^G\big). \end{aligned}$$

The coefficients $c_3(x,y)$ can be computed by the above Propositions. In particular,

$$\begin{aligned} c_3(x,x) =& \frac{-1}{6}\Big(6-12N+7N^2-N^3+(10-12N+3N^2)d_x^G \\ &+(5-3N)(d_x^G)^2+(d_x^G)^3+\eta_{\blacktriangle}^{G^c;G,G}(x)\Big). \end{aligned}$$

We may expand the heat kernel on K as follows:

$$H_t^K(x,y) = \begin{cases} 1-(N-1)t+\frac{1}{2}N(N-1)t^2-\frac{1}{6}N^2(N-1)t^3 \\ +\frac{1}{24}N^3(N-1)t^4+O(t^5) & x=y, \\ \\ t-\frac{1}{2}Nt^2+\frac{1}{6}N^2t^3-\frac{1}{24}N^3t^4+O(t^5), & x\neq y. \end{cases}$$

The second method to compute the expansion of the heat kernel is using the heat equation. By Proposition 9, we can write

$$H_t^G(x,y) := H_t^K(x,y)\big(a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+a_3(x,y)t^3+\cdots\big).$$

We remark that this method is not completely independent of the previous one, which guarantees that such an expansion is valid on some open interval containing 0.

Proposition 12 For any $x, y \in V_0$, the coefficients $a_i(x,y)$, $i = 0,1,2,3$ are as follows:

(a)

$$a_0(x,y) = \begin{cases} 1, & x = y, \\ 1, & x \neq y \text{ and } x \underset{G}{\sim} y, \\ 0, & x \neq y \text{ and } x \underset{G^c}{\sim} y, \end{cases}$$
$$= c_0(x,y).$$

(b)

$$a_1(x,y) = \begin{cases} N - 1 - d_x^G, & x = y, \\ \frac{1}{2}\left(N - d_x^G - d_y^G + \eta^{G,G}(x,y)\right), & x \neq y \text{ and } x \underset{G}{\sim} y, \\ \frac{1}{2}\eta^{G,G}(x,y), & x \neq y \text{ and } x \underset{G^c}{\sim} y, \end{cases}$$
$$= c_1(x,y).$$

(c)

$$
\begin{aligned}
&a_2(x,y)\\
&=\begin{cases}
\frac{1}{2}\Big(N^2-3N+2+(3-2N)d_x^G+(d_x^G)^2\Big), & \text{if } x=y,\\[2ex]
\frac{1}{12}\Bigg(N^2-3Nd_x^G+2(d_x^G)^2+(2-3N)d_y^G+2(d_y^G)^2 & \\
\quad +2d_x^G d_y^G+(3N-2d_x^G-2d_y^G)\eta^{G,G}(x,y) & \\
\quad -2\sum_{w\underset{G}{\sim}x,\, w\neq y,\, w\underset{G}{\sim}y} d_w^G+2\sum_{w\underset{G}{\sim}x,\, w\neq y}\eta^{G,G}(w,y)\Bigg), & \text{if } x\neq y \text{ and } x\underset{G}{\sim}y,\\[2ex]
\frac{1}{12}\Bigg((3N-2d_x^G-2d_y^G)\eta^{G,G}(x,y) & \\
\quad -2\sum_{w\underset{G}{\sim}x,\, w\neq y,\, w\underset{G}{\sim}y} d_w^G+2\sum_{w\underset{G}{\sim}x,\, w\neq y}\eta^{G,G}(w,y)\Bigg), & \text{if } x\neq y \text{ and } x\underset{G}{\not\sim}y.
\end{cases}
\end{aligned}
$$

5. Heat kernel of Laplacian on 1-forms on subgraphs of a complete graph

Recall that the number of edges in K is equal to $\#V_1 = N(N-1)/2$. Define six $(0,1)$-matrices A_i, $i = 1, \ldots, 6$, of order $\#V_1 \times \#V_1$ as follows. The rows and columns of each A_i are labeled by the edges in V_1. For $A_1, \ldots, A_6$, entries equal to 1 are, respectively, (w_{ij}, w_{ij}), (w_{ij}, w_{ji}), (w_{ij}, w_{jk}), (w_{ij}, w_{kl}), (w_{ij}, w_{ik}), (w_{ij}, w_{kl}).

By using the matrices A_i and (6), we can write the heat kernel (matrix) on K as

$$H_t^K = \sum_{i=1}^{6} u_i(t) A_i = \sum_{i=1}^{6} \frac{e^{-(N+2)t}(1+\widetilde{u}_i(t))}{N(N-1)} A_i. \tag{10}$$

Proposition 13 For all $x, y \in V_1$ and $t \geq s \geq 0$,

$$\begin{aligned} H_t^G(x,y) &= H_t^K(x,y) - \int_0^t \left(H_{t-s}^K(\Delta_1^{G^c} + L_0) H_s^G\right)(x,y)\, ds, \\ &= H_t^K(x,y) - \frac{e^{-(N+2)t}}{N(N-1)} \sum_{i=1}^{6} \int_0^t e^{(N+2)s}\left(1+\widetilde{u}_i(t-s)\right) \\ & A_i(\Delta_1^{G^c} + L_0) H_s^G(x,y)\, ds, \end{aligned}$$

where $\widetilde{u}_i(t)$ is bounded on $[0,\infty)$ and $\widetilde{u}_i(t) \to 0$ as $t \to \infty$.

Let $\mathcal{F}$ be the vector space of all real-valued functions on $[0,\infty)\times V_1$. Let $T:\mathcal{F}\to\mathcal{F}$ be a linear operator defined as

$$\begin{aligned} Tu(t,x) &:= -\int_0^t \big(H^K_{t-s}(\Delta_1^{G^c}+L_0)\big) f_s(x)\,ds \\ &= \frac{-e^{-(N+2)t}}{N(N-1)} \sum_{i=1}^{6} \int_0^t e^{(N+2)s}\big(1+\widetilde{u}_i(t-s)\big) \\ &\quad A_i(\Delta_{1,z}^{G^c}+L_{0,z})u(s,x)\,ds. \end{aligned}$$

Proposition 14 There exists a constant $\gamma>0$ such that for all $t\in(0,\gamma)$, $\|T\|<1$.

Let γ as in Proposition 14. Then by Propositions 13 and 14, for all $t \in (0, \gamma)$, we have

$$\begin{aligned} H_t^G(x,y) &= H_t^K(x,y) + TH_t^G(x,y) \\ &= H_t^K(x,y) + T(H_t^K(x,y) + TH_t^G(x,y)) \\ &= \cdots \\ &= (I + T + T^2 + \cdots)H_t^K(x,y). \end{aligned}$$

For 1-forms I cannot obtain an analogous expression for the subgragh heat kernel as that one in Corollary 8. The main reason is that for 0-forms, $TH_t^K(x,y)$ can be expressed explicitly as in Proposition 7. For 1-forms the expression is not so explicit and involves an integral, e.g., for $m=1$,

$$TH_t^K(x,y) := \frac{-e^{-(N+2)t}}{N(N-1)} \sum_{i=1}^{6} \int_0^t e^{(N+2)s} \big(1+\widetilde{u}_i(t-s)\big) \\ A_i(\Delta_{1,z}^{G^c} + L_{0,z}) H_s^K(x,y)\, ds.$$

6. Green's function of a subgraph of a complete graph

Let $K = (V, E)$ be a directed complete graph with N vertices, where $V = V^K$ is the set of all vertices and $E = E^K$ is the set of all edges. For any subgraph G of K, we let V^G and E^G denote the set of vertices and edges of G in K. We denote by G^c the complement of G in K defined as follows. Let

$$V^{G^c} := V \backslash V^G$$

and call it the set of all vertices of G^c. Also, let

$$E^{G^c} := E \backslash E^G.$$

We remark that G^c is not necessarily a graph, since an edge in E^{G^c} does not necessarily connect two vertices in V^{G^c}. For $x, y \in G$, we denoted by $x \underset{G}{\sim} y$ is there exists an edge in E^G from x to y. Similarly, we define $x \underset{G^c}{\sim} y$. Let $x \in G$ (resp. $x \in V^{G^c}$), we will abuse notation slightly be writing $x \in G$ (resp. $x \in G^c$), and denoting the cardinality $\# V^G$ (resp. $\# V^{G^c}$) simply by $\# G$ (resp. $\# G^c$).

We describe a formula for the heat kernel of a subgraph G that is obtained in Proposition 7, we first recall that the combinatorial Laplacian Δ on a graph is defined as $\Delta = A - D$, where A is the adjacency matrix and D is the degree matrix. Let Δ^K and Δ^G be the combinatorial Laplacians on K and G respectively, and let Δ^{G^c} be defined by the equation

$$\Delta^K = \Delta^G + \Delta^{G^c}.$$

In fact, Δ^G is equal to the combinatorial Laplacian on G if we regard the vertices of G^c as vertices in G that are not connected by any edge in G.

Denote the restriction of the Laplacian Δ^G to G by $\Delta^G|_G$. We comment on the relationship between $\Delta^G|_G$ and the Laplacian L and $\mathcal{L}$ defined in [Chung, 1997]. The *combinatorial Laplacian L* on the graph G is defined as

$$L(x,y) := \begin{cases} d_x^G, & x = y, \\ \\ 0, & \text{otherwise }, \\ \\ -1, & \text{if } x \neq y \text{ and } x \underset{G}{\sim} y. \end{cases}$$

Consequently,

$$\Delta^G|_G = -L.$$

We now describe a formula for the heat kernel of a subgraph G that is obtained by the authors in [Lin, Ngai and Yau, 2021]. For each $y \in V$, let $u_y^{G^c} : V \to V$ be a function defined as

$$u_y^{G^c}(x) := \begin{cases} d_x^{G^c}, & \text{if } x = y, \\ 0, & \text{if } x \neq y \text{ and } x \underset{G}{\sim} y, \\ -1, & \text{if } x \neq y \text{ and } x \underset{G^c}{\sim} y, \end{cases} \tag{11}$$

where $d_x^{G^c}$ is the number of neighbors of x in G^c.

Our goal is to compute the Green function for Δ^G. To this end, we first obtain the following formal series expansion for $\mathcal{G}^G(x,y)$, where $x, y \in G$:

$$\mathcal{G}^G(x,y) = \begin{cases} \frac{1}{N(\#G)} + \sum_{m=1}^{\infty} \left(\left(\frac{1}{\#G} - \frac{1}{N} \right) N^m - (-\Delta_x^{G^c})^{m-1} u_y^{G^c}(x) \right) \frac{1}{N^{m+1}}, & y \neq x, \\ \left(\frac{1}{\#G} - 1 \right) \frac{1}{N} + \sum_{m=1}^{\infty} \left(\left(\frac{1}{\#G} - \frac{1}{N} \right) N^m - (-\Delta_x^{G^c})^{m-1} u_y^{G^c}(x) \right) \frac{1}{N^{m+1}}, & y = x. \end{cases} \tag{12}$$

For $x, y \in G^c$, we define $\mathcal{G}^G(x,y) := 0$. For convenience we assume that $\alpha := \#V^G$ and $\beta := \#V^{G^c}$. We obtain

$$\mathcal{G}^G(x,y) = \begin{cases} \int_0^\infty \left(\frac{1}{\alpha} - H_t^G(x,y) \right) dt, & x, y \in G, \\ 0, & \text{otherwise.} \end{cases}$$

We first consider subgraphs G that are complete.

Theorem 15 Let G be a complete subgraph of a complete graph K with N vertices and let $u_y^{G^c}(x)$ be defined as in (11). Then the Green function on G is given by

$$\mathcal{G}^G(x,y) := \begin{cases} \frac{1}{(\#G)^2}, & \text{if } x, y \in G \text{ and } y \neq x, \\ \frac{1-\#G}{(\#G)^2}, & \text{if } x, y \in G \text{ and } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, both series

$$\mathcal{G}^G(x,y) = \sum_{m=1}^{\infty} \Big[\Big(\frac{1}{\#G} - \frac{1}{N} \Big) N^m - (-\Delta_x^{G^c})^{m-1} u_y^{G^c}(x) \Big] \frac{1}{N^{m+1}} + \frac{1}{N(\#G)} \qquad \text{for } y \neq x,$$

and

$$\mathcal{G}^G(x,y) = \sum_{m=1}^{\infty} \Big(\Big(\frac{1}{\#G} - \frac{1}{N} \Big) N^m - (-\Delta_x^{G^c})^{m-1} u_y^{G^c}(x) \Big) \frac{1}{N^{m+1}} + \Big(\frac{1}{\#G} - 1 \Big) \frac{1}{N} \qquad \text{for } y = x,$$

converge if and only if $x, y \in G$.

To case when G not complete is more complicated. We need to show that the term $(1/\#G - 1/N)N^m$ in the series in (12) is canceled by a corresponding term in $(\Delta_x^{G^c})^{m-1}u_y^{G^c}(x)$. Our approach is to expand $u_y^{G^c}$ in terms of the eigenfunctions of $(\Delta_x^{G^c})^{m-1}$. We first analyze the eigenspace corresponding to the eigenvalue $-N$. It turns out that if $x, y \in G$, the quantity $(\Delta_x^{G^c})^{m-1}u_y^{G^c}$ contains a term the $(1/\#G - 1/N)N^m$, along with some higher order terms. This allows us to prove the following main result.

Theorem 16

Let G be a connected (but not necessarily complete) subgraph of a complete graph K.

(a) For all $(x, y) \in G$, both series defining the Green function in (12) converge.

(b) If $x \in G^c$ or $y \in G^c$, then $\mathcal{G}^G(x, y) = 0$.

Thanks For Your Attention !