Dirichlet forms and Laplacians on fractals

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From Dirichlet form to Laplacian

- X: locally compact metric space (*fractals*, *graphs*, *manifolds*)
  μ: a σ-finite Borel measure on X

  $(\mathcal{E}, \mathcal{F})$: a Dirichlet form on X: *non-negative, densely defined, symmetric bilinear form on $L^2(X, \mu)$, closed, Markovian*

  Eg: Rie manifold $\mathcal{M}$, $\quad \mathcal{E}(u, v) = \int_{\mathcal{M}} \nabla u \cdot \nabla v d\mu, \quad \mathcal{F} = W^{1,2}(\mathcal{M})$

  weighted graph $(V, \sim)$, $\mathcal{E}(u, v) = \sum_{x \sim y} c_{x,y}(u(x) - u(y))(v(x) - v(y))$

- Δ_μ: a Laplacian on (X, μ), infinitesimal generator of $(\mathcal{E}, \mathcal{F})$

  For $u \in \mathcal{F}, f \in L^2(X, \mu)$: (Gauss-Green's formula)

$$\Delta_\mu u = f \iff \mathcal{E}(u, v) = -\int_K f v d\mu, \quad \forall v \in \mathcal{F}$$

Analysis on fractals, graphs, manifolds

On Riemannian manifolds with non-negative Ricci curvature:

- Gaussian estimate (*Li-Yau, 86', Acta. Math.* time≈dist^2):
  $p_t(x,y) \simeq \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right), \quad \forall x,y \in \mathcal{M}, t>0$ (HK$_2$)

On fractals including S. gasket and S. carpet:

- sub-Gaussian estimate (*Barlow-Perkins, Barlow-Bass, 90s'*, time≈dist$^\beta$):
  $p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d^\beta(x,y)}{ct}\right)^{1/(\beta-1)}\right), \quad \forall x,y \in K, t>0$ (HK$_\beta$)

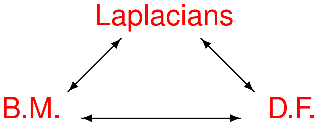
  α-Hausdorff dimension; β-walk dimension, $\alpha+1 \geq \beta \geq 2$

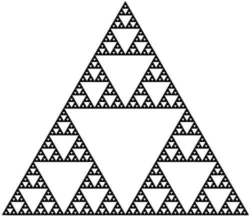
Analysis on fractals

K: a self-similar set

μ: a Radon measure on K (self-similar)

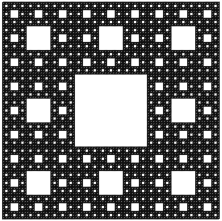
- the existence and uniqueness of Δ_μ on K
- heat kernel estimates on K





Sierpinski gasket

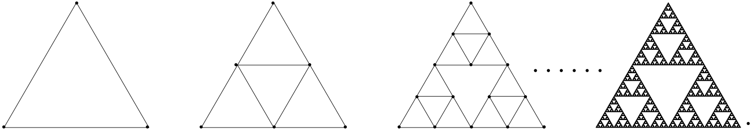
vs



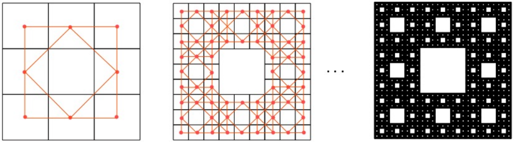
Sierpinski carpet

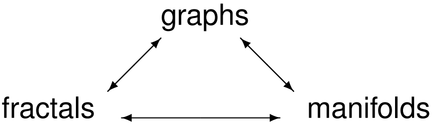
Analysis on fractals: graph approximations

S.gasket:



S.carpet:



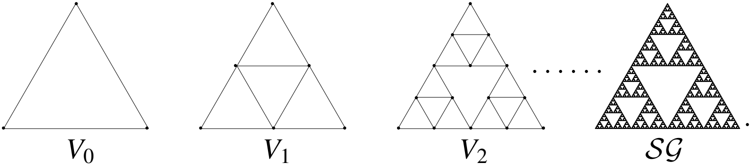


History: the Sierpinski gasket $\mathcal{SG}$

- S. gasket: $\mathcal{SG}$
  $V_0 = \{q_0, q_1, q_2\}$
  $F_i : \mathbb{R}^2 \to \mathbb{R}^2, i = 0, 1, 2$
  $F_i(x) = \frac{1}{2}(x - q_i) + q_i$

  $$\mathcal{SG} = F_0\mathcal{SG} \cup F_1\mathcal{SG} \cup F_2\mathcal{SG} \quad \text{(self-similar identity)}$$

  $V_m = \bigcup_i F_i V_{m-1}$
  $V_0 \subset V_1 \subset V_2 \subset \cdots$
  $\mu : \mu(C) = \frac{1}{3^m}$ for each m-cell C (self-similar measure)



History: $\mathcal{SG}$ vs I

- Unit interval: I
  $V_0 = \{q_0, q_1\}$
  $F_i : \mathbb{R} \to \mathbb{R}, i = 0, 1$
  $F_i(x) = \frac{1}{2}(x - q_i) + q_i$

$$I = F_0 I \cup F_1 I \qquad \text{(self-similar identity)}$$

  $V_m = \bigcup_i F_i V_{m-1}$
  $V_0 \subset V_1 \subset V_2 \subset \cdots$
  $\mathcal{L}$: $\mathcal{L}(C) = \frac{1}{2^m}$ for each m-cell C (Lebesgue measure)

History: $\mathcal{SG}$ vs I

- r : the renormalization factor
  $r = \frac{1}{2}$ for I; $r = \frac{3}{5}$ for $\mathcal{SG}$
- D.F.: energy form $(\mathcal{E}, \mathcal{F})$, limit of (rescaled) graph energies
  $\mathcal{F} := \{u \in C(\mathcal{SG}$ or $I) : \mathcal{E}(u) := \mathcal{E}(u,u) < \infty\}$
  $\mathcal{E}(u,v) := \lim_{m\to\infty} r^{-m} \sum_{x\sim_m y} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big), \quad u, v \in \mathcal{F}$
- For I, $r = \frac{1}{2}$

$$\mathcal{E}(u,v) = \lim_{m\to\infty} r^{-m} \sum_{k=1}^{2^m} \Big(u(\frac{k}{2^m}) - u(\frac{k-1}{2^m})\Big) \cdot \Big(v(\frac{k}{2^m}) - v(\frac{k-1}{2^m})\Big)$$

$$= \lim_{m\to\infty} \frac{1}{2^m} \sum_{k=1}^{2^m} \frac{u(\frac{k}{2^m}) - u(\frac{k-1}{2^m})}{1/2^m} \cdot \frac{v(\frac{k}{2^m}) - v(\frac{k-1}{2^m})}{1/2^m}$$

$$= \int_0^1 u'(x)v'(x)dx$$

History: the Laplacian on $\mathcal{SG}$

- There is a local regular D.F. $(\mathcal{E}, \mathcal{F})$ on $\mathcal{SG}$, satisfying

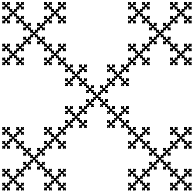
$$\mathcal{E}(u) = \frac{5}{3} \cdot \sum_{i=1}^{3} \mathcal{E}(u \circ F_i)$$

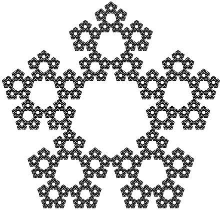
- The B.M. on $\mathcal{SG}$ (*Kusuoka, 87', Goldstein, 87', Barlow-Perkins, 88', PTRF*)—-a probabilistic approach
- The D.F. on $\mathcal{SG}$ (*Kigami, 89', Japan J. Appl. Math.*), extended to p.c.f. sets (*93', Trans. AMS*)—-an analytic approach

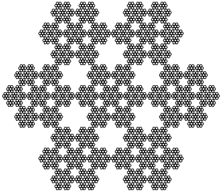
  key: to find a nonlinear fixed eigenform

History: from $\mathcal{SG}$ to p.c.f. fractals

- On nested fractals:
  existence (*Lindstrøm, 90', Mem. AMS*)
  uniqueness (*Sabot, 97', Ann. Sci. École Norm. Sup.*)
- existence problem on p.c.f. sets is still open, less progress
- Heat kernel estimates:
  on nested fractals (*Kumagai, 93', PTRF*)
  on p.c.f. sets (*Hambly-Kumagai, 99', PLMS*)


Vicsek set


pentagasket


snowflake

- Consider a rational map

$$R_{\lambda,n,m}(z) = z^n + \frac{\lambda}{z^m}, \quad n \geq 2, m \geq 1, \lambda \in \mathbb{C} \setminus \{0\}.$$

Call $R_{\lambda,n,m}$ a Misiurewicz-Sierpinski map if:

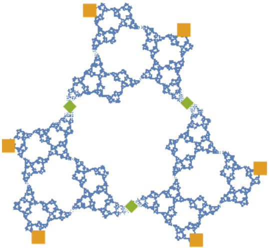
(MS1). *each critical point of $R_{\lambda,n,m}$ is strictly preperiodic*;

(MS2). *each critical point of $R_{\lambda,n,m}$ is on the boundary of the immediate attracting basin of ∞.*

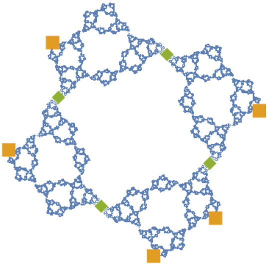
Critical points $C := \{z \in \mathbb{C} : R'_{\lambda,n,m}(z) = 0\}$

MS-Julia sets

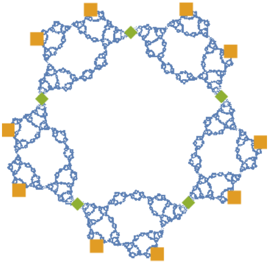
- Let $K_{\lambda,n,m}$ be the Julia set of $R_{\lambda,n,m}$.



$m=1, n=2$
$\lambda \approx -0.0380 + 0.4262i$



$m=2, n=2$
$\lambda \approx -0.0196 + 0.2753i$



$m=3, n=2$
$\lambda \approx -0.0129 + 0.2041i$

$C = \{\text{green points}\}$; $V_0 = \{\text{orange points}\}$, forward orbits of C

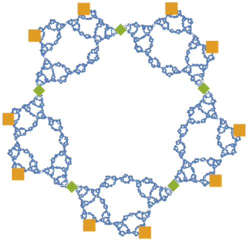
MS-Julia sets

- The dynamics and topologies of K_λ (*Devaney-Look, 06', Topology Proc.*)

  $\#C = m + n$
  $e^{\frac{2\pi i}{m+n}} C = C$, *rotational symmetric*
  C disconnects K_λ into $m + n$ components

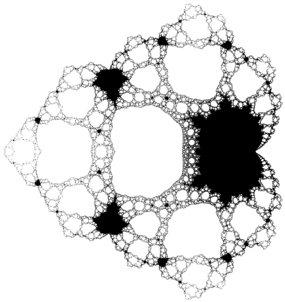
  β_λ: boundary of the attracting basin of ∞, *a simple closed curve*
  τ_λ: boundary of the trap door

  $C = \beta_\lambda \cap \tau_\lambda \Longrightarrow$ a "ring"-shape of K_λ



MS-Julia sets

- For fixed m, n, the MS λ's are dense in boundary of the locus of connectedness of R_λ(*L. Tan, 98', Nonlinearity*).
- For fixed m, n, infinitely many K_λ's are not topologically equivalent (*Devaney-Rocha-Siegmund, 07', Topology Proc.*).



The Mandelbrot set of $R_{\lambda,2,2}$

From p.c.f. fractals to MS-Julia sets

K_λ has a p.c.f. structure

- $V_0 = \bigcup_{k=1}^{\infty} R_\lambda^k(C)$, $\#V_0 < \infty$
- $\{F_i\}_{i=1}^{m+n}$, branches of R_λ^{-1}
- $F_i K_\lambda \cap F_j K_\lambda \subset C$ (at most one intersection point)
- $F_w K_\lambda \cap F_{w'} K_\lambda$, finite but complicated, depends on R_λ

Theorem 1. (Cao, Hassler, Q., Sandine, Strichartz, Adv. Math., 2021')

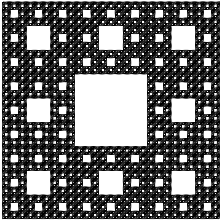
There exists a unique self-similar D.F. $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ on K_λ and $0 < r < 1$,

$$\mathcal{E}_\lambda(u) = r^{-1} \cdot \sum_{i=1}^{m+n} \mathcal{E}_\lambda(u \circ F_i), \quad \forall u, v \in \mathcal{F}_\lambda.$$

In addition, $(\mathcal{E}_\lambda, \mathcal{F}_\lambda)$ is rotational symmetric.

- analytic approach; r is unknown in general.

History: The Laplacian on $\mathcal{SC}$



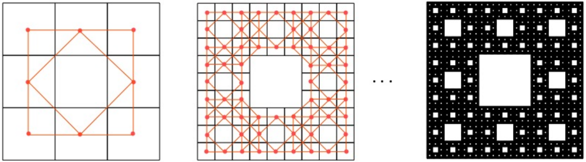
- A local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on the S. carpet $\mathcal{SC}$,

$$\mathcal{E}(u) = r^{-1} \cdot \sum_{i=1}^{8} \mathcal{E}(u \circ F_i), \quad 0 < r < 1.$$

- The existence on $\mathcal{SC}$
  B.M.: *Barlow-Bass, 89', 90', AIHP, etc.*—-a probabilistic approach
  D.F.: *Kusuoka-Zhou, 92', PTRF*—-a "nearly" analytic approach
- r is unknown!

History: From $\mathcal{SC}$ to GSCs, non-p.c.f. fractals

- cell graph approximation



- The equivalence of the two approaches (*Barlow-Bass-Kumagai-Teplyaev, 2010', JEMS*)
- heat kernel estimates (*Barlow-Bass, 92', PTRF*)
- extension to higher dimensional, GSCs (*Barlow-Bass, 99', Canad. J. Math.*), *Kusuoka-Zhou fails: from recurrent to transient.*
- Γ-convergence from non-local to local (*Grigor'yan-Yang, 2019', Trans. AMS*)

Lower bound estimate of the resistances

- A key step: $R_m \gtrsim \sigma_m (\Longrightarrow R_m \gtrsim r^{-m}, r \in (0,1))$
- K.-Z. is not purely analytic: using B.-B.'s probabilistic approach, "Knight move" and "corner move" of B.M.

$$\mathcal{D}_m(f) := \sum_{w \sim v \text{ in } W_n} \Big(⨍_{K_w} f - ⨍_{K_v} f \Big)^2, \quad m \geq 1.$$

- Poincare constants σ_m: $= \sup\limits_{n \in \mathbb{N}, w, v \in W_n, f} \frac{|⨍_{K_w} f - ⨍_{K_v} f|^2}{\mathcal{D}_{n+m, K_w \cup K_v}(f)}$.
- Resistance constants R_m: For $A, B \subset W_n$,

$$R_m(A,B) = \max \Big\{ \frac{1}{\mathcal{D}_{n+m}(f)} : f|_{K_A} = 0, f|_{K_B} = 1 \Big\}. \quad K_A = \bigcup_{w \in A} K_w$$

For $w \in W_n, n \geq 1$, write $\mathcal{N}_w$ the n-neighborhood of w.
For $m \geq 1$, $R_m = \inf\limits_{n \in \mathbb{N}, w \in W_n} R_m(w, \mathcal{N}_w^c)$.

From $\mathcal{SC}$ to USC

$\square$: the unit square in $\mathbb{R}^2$, $\mathcal{G}$: the symmetric group on $\square$.

Definition. (USC)

Let $\{F_i\}_{1\leq i\leq N}$ be a finite set of similarities with contraction ratio k^{-1}.

(Non-overlapping). $F_i(\square) \cap F_j(\square)$ is either a line segment, a point, or $\emptyset$;
(Connectivity). $\bigcup_{i=1}^N F_i(\square)$ is connected;
(Symmetry). $\bigcup_{i=1}^N F_i(\square)$ is invariant under $\mathcal{G}$;
(Boundary included). $\partial\square \subset \bigcup_{i=1}^N F_i(\square) \subset \square$.

Call the unique compact subset $K \subset \square$ satisfying

$$K = \bigcup_{i=1}^N F_i K$$

an unconstrained Sierpinski carpet (USC).

From $\mathcal{SC}$ to USC

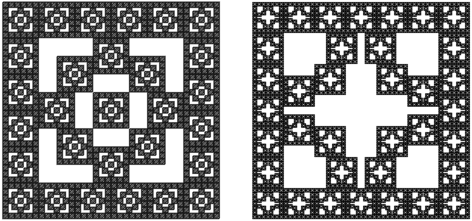


Figure: Unconstrained Sierpinski carpets (USC).

Theorem 3. (Cao and Q., 2021)

For USC, the gap $R_m \gtrsim \sigma_m$ can be fulfilled in an analytic way.

Remark.

Barlow-Bass's probabilisitic argument can not be extended to USC *(heavily depends on the local symmetry)*.

$R_m \gtrsim \sigma_m$

Three kinds of resistance constants.

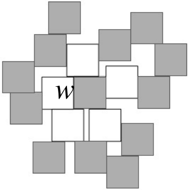
- R_m: resistance between two "concentric balls"
- $R_m^{(S)}$: resistance between opposite sides of K (or cells)

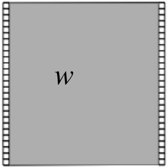
$$R_m^{(S)} = \frac{1}{\mathcal{D}_m(h_m)}, \quad \mathcal{D}_m(h_m) = \min_{f|_{K_{m,L}}=0,\ f|_{K_{m,R}}=1} \mathcal{D}_m(f)$$

- $\tilde{R}_{m,l}$: resistance between ends of chain of cells
  A: a chain of n-cells of length l, with two ends w and v

$$\tilde{R}_{m,l} = \max_{f|_{K_w}=0,\ f|_{K_v}=1} \frac{1}{\mathcal{D}_{n+m,A}(f)}.$$

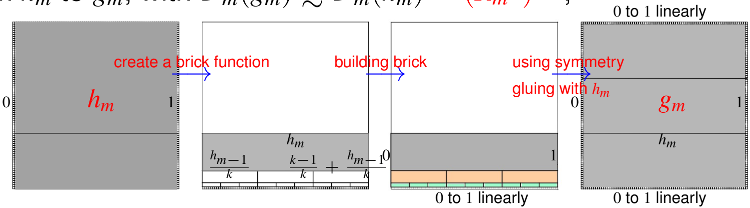
strategy: $R_m \gtrsim R_m^{(S)} \gtrsim \tilde{R}_{m,l} \gtrsim \sigma_m$



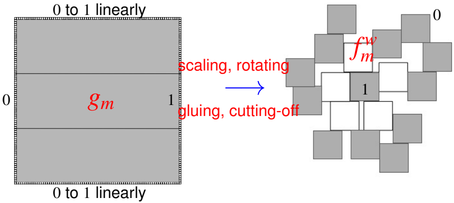


Step 1. $R_m \gtrsim R_m^{(S)}$

- from h_m to g_m, with $\mathcal{D}_m(g_m) \lesssim \mathcal{D}_m(h_m) = (R_m^{(S)})^{-1}$;



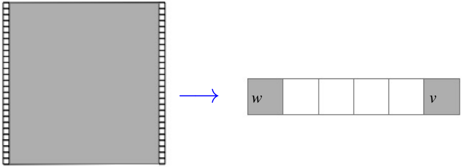
- from g_m to f_m^w, with $(R_m(w, \mathcal{N}_w^c))^{-1} \le \mathcal{D}_m(f_m^w) \lesssim \mathcal{D}_m(g_m) = (R_m^{(S)})^{-1}$.



So $R_m = \inf_{n\in\mathbb{N}, w\in W_n} R_m(w, \mathcal{N}_w^c) \gtrsim R_m^{(S)}$.

- $R_m^{(S)} \geq \frac{1}{4l} \cdot \tilde{R}_{m,l}$.

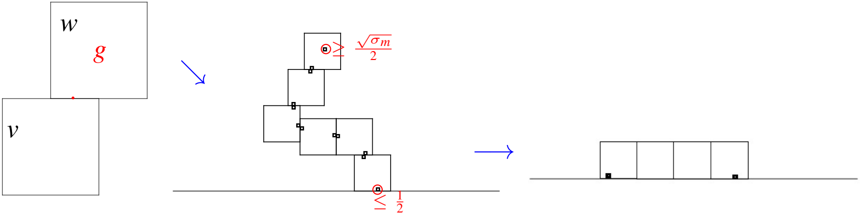
- series rule of resistances;
- for $l = 2, 3$, it is easy; for large l, using induction and symmetry.



Step 3. $\tilde{R}_{m,l} \gtrsim \sigma_m$

- For large m, there exists $m' \ll m$ independent of m, such that $\tilde{R}_{m-m',l} \gtrsim \sigma_m$, where l is bounded above depending only on m'.

- recall $\sigma_m = \sup_{n\in\mathbb{N}, w,v\in W_n, f} \frac{|⨍_{K_w} f - ⨍_{K_v} f|^2}{\mathcal{D}_{n+m,K_w\cup K_v}(f)}$;
- choose $n, w \overset{n}{\sim} v$, g' with $\mathcal{D}_{n+m,K_w\cap K_v}(g') = 1$ and $⨍_{K_w} g' - ⨍_{K_v} g' = \sqrt{\sigma_m}$;
- look at $g = g' \circ F_w$, may assume $⨍_K g = \frac{1}{2}\sqrt{\sigma_m}$, $\mathcal{D}_m(g) \le \frac{1}{2}$;
- find two $(n-m')$-cells arranged in a line with difference of g between ends comparable with $\sqrt{\sigma_m}$. $\Longrightarrow \tilde{R}_{m-m',l} \gtrsim \frac{|g(u)-g(v)|^2}{\mathcal{D}_m(g)} \gtrsim \sigma_m$.



$R_m \gtrsim \sigma_m$

m large enough, m' independent of m

- Step 1: $R_m \gtrsim R_m^{(S)}$
- Step 2: $R_m^{(S)} \gtrsim \tilde{R}_{m,l}$
- Step 3: $\tilde{R}_{m-m',l} \gtrsim \sigma_m$

- Observation: $\sigma_m \asymp \sigma_{m-m'}$

Combing Steps 1-3, we have $R_{m-m'} \gtrsim \sigma_m$, which gives $R_m \gtrsim \sigma_m$.

Let K be a USC and μ be the normalized Hausdorff measure on K.

Theorem 4. (S. Cao and Q., 2021)

There exists a unique local regular symmetric self-similar D.F. $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$, satisfying

$$\mathcal{E}(u) = r^{-1} \cdot \sum_{i=1}^{8} \mathcal{E}(u \circ F_i), \quad 0 < r < 1.$$

Also, the sub-Gaussian HK estimate holds.

From $\mathcal{SC}$ to USC

Let K, K_n, $n \geq 1$ be USC with fixed k, $(X, \mathbb{P}_x)$, $(X^{(n)}, \mathbb{P}_x^{(n)})$, $n \geq 1$ be the associated B.M..

Theorem 5. (S. Cao and Q., 2021)

$\mathbb{P}_{x_n}^{(n)}\big((X_t^{(n)})_{t\geq 0} \in \cdot\big) \Rightarrow \mathbb{P}_x\big((X_t)_{t\geq 0} \in \cdot\big)$, $\forall x_n \to x \iff$ the geodesic metrics $d_{G,n}$ on K_n, $n \geq 1$ are equicontinuous.

$(K_n, R_n) \to (K, R)$ in the sense of Gromov-Hausdorff convergence

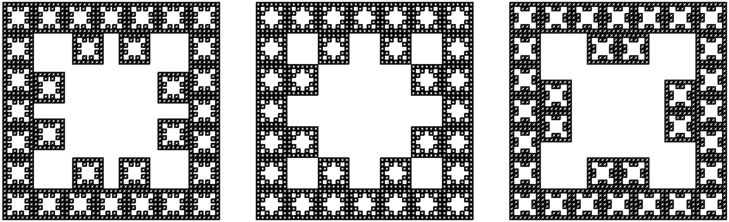


Figure: The USC $K(z)$.

From USC to LSC

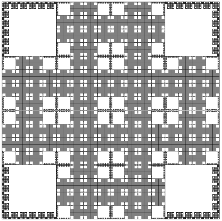
- Sierpinski carpet like fractals (LSC)

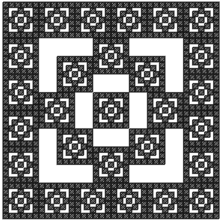
Definition. (LSC)

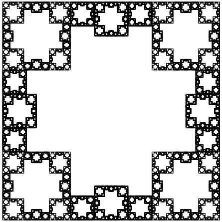
Let $N \geq 8$, and for $1 \leq i \leq N$, choose $F_i : \square \to \square$ of the form

$$F_i(x) = \rho_i x + c_i.$$

The unique compact K satisfying $K = \bigcup_{i=1}^{N} F_i K$ is a Sierpinski carpet like fractal (LSC) if *(Non-overlapping)*, *(Connectivity)*, *(Symmetry)*, *(Boundary included)* are satisfied.



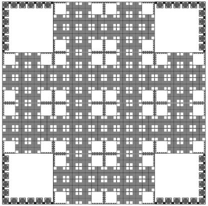




From USC to LSC

Question.

Does there always exist a D.F. with sub-Gaussian HK estimate on a LSC? If not, when?



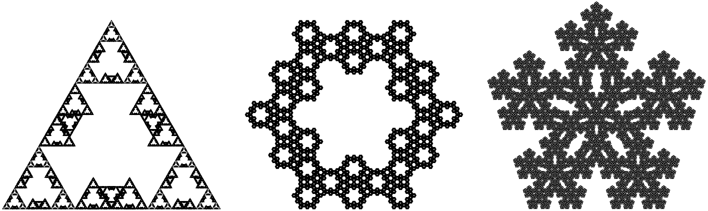
Theorem 6. (Cao and Q., 2023, PAMS)

There does not exist a D.F. with sub-Gaussian HK estimate on the $\mathcal{LSC}$ pictured above.

Partially answer to when: there exist such D.F.s on USC and uncountably many hollow LSC.

From USC to LSC

- Polygon carpets (Cao, Q., Wang, 2022)
  —-analytic approach extending $2d$-USC



- Higher dimensional USC (Cao, Q., 2023)
  —-a probabilistic approach using coupling

Thank you!