Mandelbrot Cascades

Critical moments,Rajchman measures and Sobolev smoothness

Yanqi Qiu
joint with Xinxin Chen, Yong Han and Zipeng Wang

March 23, 2024

Mandelbrot's classical setting

- Let $b \in \mathbb{N}$, for $n = 1, 2, \ldots$,$k \in \{1, 2, \ldots, n\}$, define intervals:

$$I\left(j_1, j_2, \ldots, j_n\right) = \left[\sum_{k=1}^{n} \frac{j_k}{b^k}, \sum_{k=1}^{n} \frac{j_k}{b^k} + b^{-n}\right) \subset [0, 1],$$

  with $j_i \in \{0, 1, \ldots, b-1\}$.
- W is a positive random variable (r.v.) with $\mathbb{E}(W) = 1$.
- $W\left(j_1, j_2, \ldots, j_n\right)$ be independent copies of W.
- Let μ_n be the measure defined on $[0, 1]$, whose density on the interval $I\left(j_1, j_2, \ldots, j_n\right)$ is given by

$$W\left(j_1\right) W\left(j_1, j_2\right) \ldots W\left(j_1, j_2, \ldots, j_n\right).$$

Mandelbrot's classical setting

- The total mass of μ_n is given by

$$Y_n = \|\mu_n\| = b^{-n} \sum_{j_1, j_2, \ldots, j_n} W(j_1) W(j_1, j_2) \ldots W(j_1, j_2, \ldots, j_n).$$

- This is a nonnegative martingale ($\mathbb{E}(Y_n) = 1$). It converges a.s. to a r.v. Y_∞ such that $\mathbb{E}(Y_\infty) \leq 1$.
- For any b-adic intervals $I, \mu_n(I)$ is a martingale with expectation $|I|$ which converges a.s. to a limit $\mu(I)$. Hence μ_n tends weakly a.s. to a measure μ of total mass Y_∞.

Mandelbrot's classical setting

- Y_n satisfies the following equation

$$Y_n = b^{-1} \sum_{j=0}^{b-1} W(j) Y_{n-1}(j),$$

where $W(j)$ and $Y_{n-1}(j)$ are all independent, and the $Y_{n-1}(j)$ have the same distribution as Y_{n-1}.

- Informally, let $n \to \infty$, one gets the functional equation

$$\mathrm{Z} = b^{-1} \sum_{j=0}^{b-1} W_j Z_j \tag{1}$$

where the r.v.'s W_j and Z_j are all independent, the W_j having the same distribution as W, and the Z_j having the same distribution as Z.

Basic Questions

(1) (non-degeneracy) When is Y_∞ non-trivial?

(2) When $Y_n \to Y_\infty$ in L^p-norm?

Theorem (J.P.Kahane-J.Peyrière 1976)

Any one of the following 4 conditions is the necessary and sufficient condition for the non-degeneracy:

(1) $\mathbb{E}\,(Y_\infty) = 1$,

(2) $\mathbb{E}\,(Y_\infty) > 0$,

(3) *Equation* (1) *has a solution* Z *such that* $\mathbb{E}(Z) = 1$,

(4) $\mathbb{E}(W \log W) < \log b$.

Theorem (Condition for the existence of finite moments)

Let $p > 1$. One has $0 < \mathbb{E}\left(Y_\infty^p\right) < \infty$ if and only if $\mathbb{E}\left(W^p\right) < b^{p-1}$.

Further Questions

(1) If $\mathbb{E}\left(W^p\right) < b^{p-1}$, what's the convergence rate of $\mathbb{E}\left(Y_n^p\right)$?

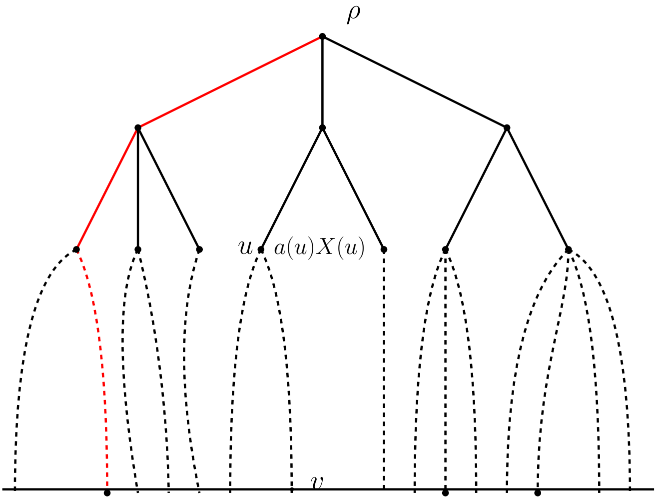
(2) If $\mathbb{E}\left(W^p\right) \geq b^{p-1}$, what's the divergence rate of $\mathbb{E}\left(Y_n^p\right)$?

We will answer these questions after a while.

History

- Mandelbrot's cascade measures
- Kahane, Peyriere 1985,1976
- Aihua Fan, Barrel ...
- Kahane's GMC: Kahane, Sheffield, Remi Rhode, etc

A general framework (the tree is not necessarily homogeneous)



$$Z_N := \Sigma_{v\in T_N} a(v)\,\Pi_{u\preceq v} X(u) = a(\rho) + \Sigma_{m=1}^{N} \Sigma_{v\in S_m} a(v)\,\Pi_{u\preceq v} X(u).$$

- $\mathcal{T}$ is a rooted (denote by ρ the root) infinite tree with the natural partial order $\succeq$ and the graph distance $d(\cdot,\cdot)$.
- For $n \geq 0$, define

$$S_n := \{v \in \mathcal{T} : d(v,\rho) = n\}\,, \quad T_n := \{v \in \mathcal{T} : d(v,\rho) \leq n\}\,.$$

Let $X(\rho) = 1$. For any positive integer N, define

$$Z_N := \sum_{v \in T_N} a(v) \prod_{u \preceq v} X(u) = a(\rho) + \sum_{m=1}^{N} \sum_{v \in S_m} a(v) \prod_{u \preceq v} X(u),$$

$X(u)$ for $u \neq \rho$ are i.i.d non-negative r.v.'s and $a(v) \geq 0$ are considered as weights.

Remark: Here no assumption $\mathbb{E}(X(u)) = 1$.

A natural question

Question

When is $(Z_N)_{N=1,2,3,\ldots}$ uniformly bounded in L^p for $p > 1$?

Theorem (Han-Q.-Wang)

For $p \geq 2$ and positive r.v. X with $\mathbb{E}[X^p] < \infty$,

$$\|Z_N\|_p^p \asymp \|\tilde{Z}_{N-1}\|_{\frac{p}{2}}^{\frac{p}{2}} \asymp \left(a(\rho) + \sum_{n=1}^{N} \sum_{v \in S_n} a(v) \left(\mathbb{E}[X]\right)^n\right)^p$$

$$+ \sum_{v \in S_1} \left(\sum_{n=1}^{N} \left(\mathbb{E}[X]\right)^{n-1} \sum_{w \in S_n, w \succeq v} a(w)\right)^p \operatorname{Var}(X)^{\frac{p}{2}}$$

$$+ \operatorname{Var}(X)^{\frac{p}{2}} \left\| \sum_{m=1}^{N-1} \sum_{v \in S_m} \sum_{\substack{x \in S_{m+1} \\ x \succeq v}} \left(\sum_{n=m+1}^{N} \left(\mathbb{E}[X]\right)^{n-m-1} \sum_{\substack{w \in S_n \\ w \succeq x}} a(w) \right)^2 \prod_{u \preceq v} X(u)^2 \right\|_{\frac{p}{2}}^{\frac{p}{2}},$$

where the first constant depend only on $\mathbb{E}[X^p]$ and p and the second constant depends on $\mathbb{E}[X^p]$ and p and the number $|S_1|$ of children of the root of the tree.

Consequences

The new random variable $\tilde{Z}_{N-1}$ has the same structure as Z_n:

$$\tilde{Z}_{N-1} = \tilde{a}(\rho) + \sum_{m=1}^{N-1} \sum_{v \in S_m} \tilde{a}(v) \prod_{u \preceq v} \tilde{X}(u),$$

where $\tilde{X}(u) = X(u)^2$ and

$$\tilde{a}(\rho) = \left(a(\rho) + \sum_{n=1}^{N} \sum_{v \in S_n} a(v) \left(\mathbb{E}[X]\right)^n \right)^2$$

$$+ \sum_{v \in S_1} \left(\sum_{n=1}^{N} \left(\mathbb{E}[X]\right)^{n-1} \sum_{w \in S_n, w \succeq v} a(w) \right)^2 \mathrm{Var}(X)$$

and for $v \in S_m (1 \leq m \leq N-1)$,

$$\tilde{a}(v) = \sum_{x \in S_{m+1}, x \succeq v} \left(\sum_{n=m+1}^{N} \left(\mathbb{E}[X]\right)^{n-m-1} \sum_{\substack{w \in S_n \\ w \succeq x}} a(w) \right)^2 \mathrm{Var}(X).$$

Main tools

Martingale inequalities

- Burkholder inequalities
- Burkholder-Rosenthal inequalities

Consequences

Corollary (Application in canonical Mandelbrot's situation)
Let N be a positive integer and $p \geq 1$. If

$$a(\rho) = 0, \quad \mathbb{E}[X] = 1.$$

and

$$a(v) = 0, \quad \forall v \notin S_N, \quad a(v) = \frac{1}{b^N}, \quad \forall v \in S_N.$$

Then $\|Z_N\|_p < \infty$ if and only if $\mathbb{E}[X^p] < b^{p-1}$. Moreover, we have

$$\begin{cases} \lim_{N\to\infty} \frac{\log \|Z_N\|_p^p}{N} = \log \frac{\mathbb{E}[X^p]}{b^{p-1}} & \mathbb{E}[X^p] > b^{p-1} \\ \lim_{N\to\infty} \frac{\log \|Z_N\|_p^p}{\log N} = 1 & \mathbb{E}[X^p] = b^{p-1}. \end{cases}$$

Aihua Fan's result

- Let $P = (p_{i,j})$ be a primitive transition matrix of a Markov chain indexed by $\{0, 1, 2, ..., b-1\}$, i.e. $P^m > 0$ for some $m \geqslant 1$.
- For any $p \in \mathbb{R}$, let $\rho(p)$ be the spectral radius of the matrix $P(p)$ defined by $\left(p_{i,j}^p\right)$.

$$P(p) = \begin{pmatrix} p_{0,0}^p & p_{0,1}^p & \cdots & p_{0,b-1}^p \\ p_{1,0}^p & p_{1,1}^p & \cdots & p_{1,b-1}^p \\ \vdots & \vdots & \ddots & \vdots \\ p_{b-1,0}^p & p_{b-1,1}^p & \cdots & p_{b-1,b-1}^p \end{pmatrix}.$$

The function $\rho(p)$ is real analytic.

Theorem (Fan, 2002)

If

$$a(v) = 0, \quad v \notin S_N; \quad a(v) = \prod_{i=0}^{N-1} P(v_i, v_{i+1}) = P^{(N)}(v).$$

Then $\sup_{N \geq 1} \|Z_N\|_p < \infty$ *for* $p > 1$ *if and only if* $\mathbb{E}[X^p]\rho(p) < 1$;

We can recover this result by our new method.

Burkholder theorem

Let $M = \{(M_i, \mathcal{F}_i) : i = 0, \ldots, n\}$ be a martingale. Define martingale increments

$$D_0(M) = M_0, D_1(M) = M_1, \quad D_i(M) = M_i - M_{i-1}, \quad i = 2, \ldots, n.$$

The quadratic variation process is defined as

$$Q_i = D_0^2(M) + \ldots D_i^2(M) \quad \text{for } i = 0, \ldots, n.$$

Theorem (Burkholder inequality)

For any $p \in (1, \infty)$, we have

$$\|M_n\|_p \asymp \left\| \sqrt{Q_n} \right\|_p .$$

where the constants depend only on p.

Burkholder-Rosenthal theorem

For $n \geq 1$, the conditional square function of M is defined by

$$s_n(M) = \left[\sum_{i=0}^{n} \mathbb{E}\left(|D_i(M)|^2 \mid \mathcal{F}_{i-1}\right)\right]^{1/2}, \quad n = 0, 1, 2, \ldots.$$

Theorem (Burkholder-Rosenthal inequality)

For any $p \in [2, \infty)$, we have

$$\|M_n\|_p \asymp \|s_n(M)\|_p + \left\| \left(\sum_{i=0}^{n} |D_i(M)|^p \right)^{1/p} \right\|_p$$

where the constants depend only on p.

Thank you for your attention!