Application of the heat kernel to Riesz transform on some gluing manifolds

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Contents

Set $V(x,R) := Vol(B(x,R))$. We say that (D) holds if

$$(D) \qquad V(x,2R) \le CV(x,R),\ \forall\, x \in X \,\&R > 0.$$

An upper Gaussian bound of the heat kernel holds, if

$$(UE) \qquad h_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left\{-\frac{d(x,y)^2}{ct}\right\}.$$

The Li-Yau estimate holds for the heat kernel if

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Definition (Poincaré inequality)

We say a Poincaré inequality (P_2) holds, if there exists $C_P > 0$ s.t. for any ball $B = B(x, r)$, and any smooth function f on B it holds

$$(P_2) \qquad \fint_B |f - f_B| \, d\mu \le C_P r \left(\fint_B |\nabla f|^2 \, d\mu \right)^{1/2}.$$

A local Poincaré inequality $(P_{2,\mathrm{loc}})$ holds if for any $r_0 > 0$, there exists $C_P(r_0)$ such that the above inequality holds for any ball with radius less than r_0.

DEFINITION (POINCARÉ INEQUALITY)

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Due to Saloff-Coste and Grigor'yan, it is well-known that on a manifold, the following conditions are equivalent:

1. (D) and (P_2) hold;
2. a parabolic Harnack inequality for the heat equation $\Delta u = \frac{\partial}{\partial t} u$ holds.
3. Li-Yau estimate holds.

Remark

(i) An important consequence of the above result, in view of (1), is that conditions (2) & (3) are invariant under quasi-isometries (biLipschitz map).
(ii) Obviously $(D) + (P_2)$ or equivalently (LY) implies (UE).

Definition

Let $p \in (2, \infty]$. We say that the quantitative reverse Hölder inequality for gradients of harmonic functions (for short, (RH_p)) holds, if for every $u \in W^{1,2}(2B)$, $B = B(x_0, R)$, satisfying $\Delta u = 0$ in $2B$, it holds

$$(RH_p). \qquad \left(\fint_B |\nabla u|^p \, d\mu \right)^{1/p} \le \frac{C}{R} \fint_{2B} |u| \, d\mu,$$

Remark

By using Caccioppli's inequality, it's easy to see that $(\widetilde{RH_p})$ implies (RH_p),

$$(\widetilde{RH_p}) \qquad \left(\fint_B |\nabla u|^p \, d\mu \right)^{1/p} \le C \fint_{2B} |\nabla u| \, d\mu$$

they are equivalent if one has (P_2). In general (RH_p) is weaker than $(\widetilde{RH_p})$.

1. Background

Definition

Let $p \in (2, \infty]$. We say that the quantitative reverse Hölder inequality for gradients of harmonic functions (for short, (RH_p)) holds, if for every $u \in W^{1,2}(2B)$, $B = B(x_0, R)$, satisfying $\Delta u = 0$ in $2B$, it holds

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Theorem (Coulhon-J.-Koskela-Sikora '20)

Assume that (D) *and* (UE) *hold on* (M, d, μ). *Then TFAE:*

(i) (RH_∞) *holds;*

(ii) (GLY_∞) *holds: i.e.,* $\exists\, C, c > 0$ *such that* $\forall\, t > 0$ *and a.e.* $x,\, y \in X$ *it holds*

$$(GLY_\infty) \qquad |\nabla_x h_t(x,y)| \le \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp\left\{-\frac{d(x,y)^2}{ct}\right\}.$$

(iii) (G_∞) *holds: i.e., for each* $t > 0$ *it holds* $\||\nabla H_t|\|_{\infty\to\infty} \le C/\sqrt{t}$.

(iv) (GBE) *holds: i.e.,* $\exists\, C, c > 0$ *such that* $\forall\, t > 0$ *and* $\forall\, f \in W^{1,2}(X)$ *and a.e.* $x \in X$ *that*

$$|\nabla H_t f(x)|^2 \le C H_{ct}(|\nabla f|^2)(x).$$

THEOREM (COULHON-J.-KOSKELA-SIKORA '20)

Assume that (D), (UE) *and* $(P_{2,\mathrm{loc}})$ *hold (in particular* (P_2)*). Let* $p_0 \in (2,\infty)$*. Then TFAE:*

(i) (RH_{p_0}) *holds;*

(ii) (GLY_{p_0}) *holds: there exists* $\gamma > 0$ *such that for each* $t > 0$ *and a.e.* $y \in X$ *it holds*

$$(GLY_{p_0}) \quad \int_X |\nabla_x h_t(x,y)|^{p_0} \exp\left\{\gamma d(x,y)^2/t\right\} d\mu(x) \le \frac{C}{t^{p_0/2}[V(y,\sqrt{t})]^{p_0-1}}.$$

(iii) (G_{p_0}) *holds: the gradient heat semigroup* $|\nabla H_t|$ *is bounded on* $L^{p_0}(X)$ *for each* $t > 0$ *with*

$$(G_{p_0}) \qquad \||\nabla H_t|\|_{p_0\to p_0} \le C/\sqrt{t}.$$

Remark

(i) A typical example where (D), (UE) and $(P_{2,\mathrm{loc}})$ hold, but not (P_2), is a Riemannian manifold, obtained by gluing finite Euclidean ends together through a compact manifold.

(ii) Suppose that $V(x,r) \lesssim r^N$ for some $N > 2$. Then for $p > N$, (RH_p) implies that if $\Delta u = 0$ in M, then

$$\left(\int_B |\nabla u|^p \, d\mu\right)^{1/p} \le \frac{C\mu(B)^{1/p}}{R} \fint_{2B} |u - u(x_B)| \, d\mu \lesssim R^{N/p-1} \fint_{2B} |u - u(x_B)| \, d\mu.$$

This implies that there is no harmonic function of growth less than $d(x, x_B)^{1-N/p}$ other than constant, and therefore, M cannot have more than one end by the results on harmonic functions of Li-Tam'92.

Let M be a non-compact, connected manifold. We simply recognize M as the union of a compact subset M_0, and finitely many ends $\{E_i\}_{1\le i\le k}$, i.e., $M = M_0 \cup \cup_i E_i$.

Doubling: (D) together with connectivity implies that there exists $0 < N < \infty$ such that

$$(D_N) \qquad \frac{V(x,R)}{V(x,r)} \lesssim \left(\frac{R}{r}\right)^N, \forall x \in M \,\&\, 1 < r < R < \infty,$$

and $n \in (0, N]$ such that for a fixed $x_M \in M_0$

$$(RD_n) \qquad \left(\frac{R}{r}\right)^n \lesssim \frac{V(x_M,R)}{V(x_M,r)}, \; 1 < r < R < \infty.$$

THEOREM (J. 2021)

Assume that (D_N) *and* (RD_n) *hold on* $M = M_0 \cup \cup_i E_i$ *with* $2 < n \le N < \infty$. *Suppose that* (UE) *holds, and the* L^2*-Poincaré inequality* (P_2) *holds for all remote balls* B *(i.e.,* $2B \cap M_0 = \emptyset$*). Let* $p_0 \in (2, n)$. *Then TFAE:*

(i) (G_{p_0}) *holds, i.e.,* $|||\nabla e^{t\Delta}|||_{p_0 \to p_0} \le \frac{C}{\sqrt{t}}$.

(ii) (GLY_{p_0}) *holds: there exists* $\gamma > 0$ *such that for each* $t > 0$ *and a.e.* $y \in X$ *it holds*

$$(GLY_{p_0}) \quad \int_X |\nabla_x h_t(x,y)|^{p_0} \exp\left\{\gamma d(x,y)^2/t\right\} d\mu(x) \le \frac{C}{t^{p_0/2}[V(y, \sqrt{t})]^{p_0-1}}.$$

(iii) (RH_{p_0}) *holds, i.e., for any harmonic function* u *on a ball* $B(x, 2r)$*, it holds*

$$(RH_{p_0}) \qquad \left(\fint_B |\nabla u|^{p_0} \, d\mu\right)^{1/p_0} \le \frac{C}{r} \fint_{2B} |u| \, d\mu.$$

2. Gluing manifolds satisfying doubling condition

Theorem (J. 2021)

Assume that (D_N) *and* (RD_n) *hold on* M *with* $2 < n \le N < \infty$. *Suppose that* (UE) *and the* L^2*-Poincaré inequality* (P_2) *holds for remote balls* B. *Let* $p \in (2, n)$. *Then TFAE:*

(i) (RH_p) *holds.*

(ii) (RH_p^E) *holds, where* (RH_p^E) *means that there exists* $C > 0$ *such that for any ball* B, *with* $3B \cap M_0 = \emptyset$, *and any harmonic function* u *on* $2B$, *it holds*

$$(RH_p^E) \qquad \left(\fint_B |\nabla u|^p \, d\mu \right)^{1/p} \le \frac{C}{r_B} \fint_{2B} |u| \, d\mu.$$

Remark

If $Ric_M(x) \ge -\frac{c_M}{[d(x,x_M)+1]^2}$, then (RH_∞^E) holds.

Theorem (J. 2021)

Let $k \geq 2$. Suppose that for each $1 \leq i \leq k$, M_i is a complete non-compact manifold where (D), (UE) and the L^2-Poincaré inequality (P_2) holds for remote balls B $(2B \cap M_0 = \emptyset)$.

Assume that the gluing manifold $M := M_1 \# \cdots \# M_k$ satisfies (D_N) and (RD_n) for some $2 < n \leq N < \infty$.

Then if for some $p \in (2, n)$, (RH_p) holds on each M_i, (RH_p) holds on M.

Theorem (J. 2021)

Assume that (D_N) holds on M with $0 < N < \infty$, and that (UE) and (P_2^E) hold. Let $p \in (N \vee 2, \infty)$. Then the following statements are equivalent.
(i) (RH_p) holds, i.e. for any ball B and any harmonic function u on $2B$, it holds

$$(RH_p) \qquad \left(\fint_B |\nabla u|^p \, d\mu\right)^{1/p} \le \frac{C}{r_B} \fint_{2B} |u| \, d\mu.$$

(ii) (G_p) holds, i.e., $\||\nabla e^{t\Delta}\||_{p\to p} \le \frac{C}{\sqrt{t}}$.
Moreover, if M is non-parabolic, then any of the conditions implies that M can have only one end.

COROLLARY (J. 2021)

Assume that (D_N) holds on M with $0 < N < \infty$, and that (UE) and the L^2-Poincaré inequality (P_2) holds for all balls B such that $2B \cap M_0 = \emptyset$. If there exists a non-constant harmonic function u on M with the growth

$$u(x) = \mathcal{O}(d(x,o)^{\alpha}) \text{ as } d(x,o) \to \infty$$

for some $\alpha \in [0,1)$ and a fixed $o \in M$, then (R_p) does not hold for any $p > N \vee 2$ satisfying $p(1-\alpha) \geq N$.

3. The non-doubling case

We next consider non-compact manifolds with ends of different volume growth, $M = M_1 \# \cdots \# M_\ell$, for instance, $\mathbb{R}^n \# (\mathbb{R}^m \times \mathcal{M}_{n-m})$.

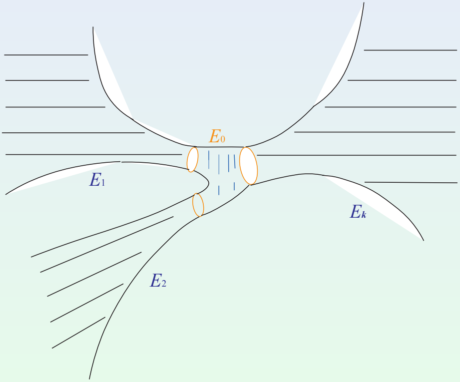

Figure: A manifold with non-doubling volume let $B(x,r)$ (resp. $B_i(x,r)$ with $1 \le i \le \ell$) denote the geodesic ball in M (resp. M_i). We set

$$|x| := \sup_{y \in E_0}\{d(x,y)\}, \quad V(x,r) := V(B(x,r)),$$

$$V_i(x,r) := V_i(B_i(x,r)), \quad V_i(r) := V_i(o_i, r),$$

where $o_i \in \partial E_i$ is a fixed reference point. Note that ∂E_i is the set that connecting $E_i = M_i \setminus K_i$ to E_0, and it always holds that

$$d(x,y) \le |x| + |y|. \tag{3.1}$$

We shall assume that for each M_i, (D) and (UE) holds.

Moreover, for some $x_i \in M_i$ and all $R \geq r \geq 1$, it holds either

$$c_i \left(\frac{R}{r}\right)^2 \leq \frac{V_i(x_i, R)}{V_i(x_i, r)} \leq C_i \left(\frac{R}{r}\right)^2,$$

or for some $n_i > 2$

$$c_i \left(\frac{R}{r}\right)^{n_i} \leq \frac{V_i(x_i, R)}{V_i(x_i, r)}.$$

Let

$$H(x,t) := \min\left\{1; \frac{|x|^2}{V_{i_x}(|x|)} + \left(\int_{|x|^2}^{t} \frac{ds}{V_{i_x}(\sqrt{s})}\right)_+\right\},$$

where $(\cdot)_+$ denotes the non-negative part and i_x denotes the index of the end that x belongs to.

Under our assumptions of the volume growth and the doubling condition,

$$H(x,t) \sim \begin{cases} \frac{|x|^2}{V_{i_x}(|x|)}, & n_{i_x} > 2 \\ 1, & n_{i_x} = 2 \end{cases}$$

Therefore, we have the uniform bound

$$H(x,t) \lesssim \frac{|x|^2}{V_{i_x}(|x|)} \lesssim 1.$$

Theorem (Grigor'yan & Saloff-Coste 2009)

Let $M = M_1 \# \cdots \# M_\ell$, where for each M_i, (D) and (UE) holds.
(i) The small time heat kernel Gaussian upper bounds hold, namely

$$h_t(x,y) \lesssim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right), \qquad \forall\, 0 < t \le 1,\ x, y \in M.$$

(ii) For $t > 1$ and $x, y \in E_i$ $(1 \le i \le \ell)$,

$$\begin{aligned} &h_t(x,y) \\ &\lesssim \left(\frac{1}{V_0(\sqrt{t})}\frac{|x|^2}{V_i(|x|)}\frac{|y|^2}{V_i(|y|)} + \frac{1}{V_i(x,\sqrt{t})}\frac{|y|^2}{V_i(|y|)} + \frac{1}{V_i(y,\sqrt{t})}\frac{|x|^2}{V_i(|x|)}\right) e^{-c\frac{|x|^2+|y|^2}{t}} \\ &+ \min\left(\frac{1}{V_i(x,\sqrt{t})}, \frac{1}{V_i(y,\sqrt{t})}\right) \exp\left\{-c\frac{d(x,y)^2}{t}\right\}, \end{aligned}$$

Theorem

(iii) For $t > 1$ and $x \in E_i, y \in E_j$, where $0 \le i, j \le \ell$ and $i \neq j$, it holds that

$$h_t(x,y) \lesssim \left(\frac{1}{V_0(\sqrt{t})} \frac{|x|^2}{V_i(|x|)} \frac{|y|^2}{V_j(|y|)} + \frac{1}{V_i(x,\sqrt{t})} \frac{|y|^2}{V_j(|y|)} + \frac{1}{V_j(y,\sqrt{t})} \frac{|x|^2}{V_i(|x|)} \right) e^{-c\frac{|x|^2+|y|^2}{t}}.$$

By the above Grigor'yan & Saloff-Coste theory, we see that if M has Ricci curvature bound, then it holds that

$$|\nabla_x h_t(x,y)| \lesssim \frac{1}{\sqrt{t}V(x,\sqrt{t})}\exp\left(-c\frac{d(x,y)^2}{t}\right), \qquad \forall\, 0<t\le 1,\ x,y\in M.$$

Qustion What about large time behavior of $|\nabla_x h_t(x,y)|$?

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Qustion What about large time behavior of $|\nabla_x h_t(x,y)|$?

Theorem (Davies)

Suppose that $\delta \in (0,1)$, $\epsilon \in (0, \frac{1}{8})$, $x, y \in M$ *and* $t > 0$. *Let* a, b, c *be positive constants such that* $c \in (0,1]$, *and that*

$$h_{(1-\delta)t}(x,x) \le a, \quad h_{(1-\delta)t}(y,y) \le b, \quad |h_s(x,y)| \le \sqrt{ab}c$$

for all $s \in ((1-\delta)t, (1+\delta)t)$. *Then for any* $m \in \mathbb{N}$, *it holds*

$$\left|\frac{\partial^m}{\partial t^m} h_t(x,y)\right| \le \frac{m!}{(\epsilon\delta t)^m} \sqrt{ab}c^{1-3\epsilon}.$$

The case $i = j$

Set

$$K(x,t) = \max\left\{\frac{C}{V_{i_x}(x,\sqrt{t})}, \frac{|x|^2}{V_0(\sqrt{t})V_{i_x}(|x|)^2}\right\}.$$

Then

$$h_t(x,y) \lesssim \sqrt{K(x,t)K(y,t)}\exp\left(-\frac{d(x,y)^2}{ct}\right),$$

which together with Davies' theorem implies that

$$t|\partial_t h_t(x,y)| \lesssim \sqrt{K(x,t)K(y,t)}\exp\left(-\frac{d(x,y)^2}{ct}\right).$$

Therefore, if $Ric_M(x) \geq -\frac{C_M}{[d(x,x_M)+1]^2}$, then

$$|\nabla_x h_t(x,y)| \lesssim \frac{1}{\sqrt{t}\wedge|x|}\sqrt{K(x,t)K(y,t)}\exp\left(-\frac{d(x,y)^2}{ct}\right)$$

The case $i \neq j$

$$t|\partial_t h_t(x,y)| \lesssim (K(x,t)K(y,t))^{3\epsilon/2}\left[\frac{\left(\frac{H(x,t)H(y,t)}{V_0(\sqrt{t})}+\frac{H(y,t)}{V_{i_x}(\sqrt{t})}+\frac{H(x,t)}{V_{i_y}(\sqrt{t})}\right)}{\exp\left(\frac{|x|^2+|y|^2}{ct}\right)}\right]^{1-3\epsilon}$$

Therefore, if $Ric_M(x) \geq -\frac{C_M}{[d(x,x_M)+1]^2}$, then

$$|\nabla_x h_t(x,y)| \lesssim \frac{(K(x,t)K(y,t))^{3\epsilon/2}}{\sqrt{t}\wedge|x|}\left[\frac{\left(\frac{H(x,t)H(y,t)}{V_0(\sqrt{t})}+\frac{H(y,t)}{V_{i_x}(\sqrt{t})}+\frac{H(x,t)}{V_{i_y}(\sqrt{t})}\right)}{\exp\left(\frac{|x|^2+|y|^2}{ct}\right)}\right]^{1-3\epsilon}$$

What we expect is: Let $n_0 := \min\{n_i\}$. Recall that $n_i \geq 2$ satisfying

$$c_i \left(\frac{R}{r}\right)^{n_i} \leq \frac{V_i(x_i, R)}{V_i(x_i, r)}.$$

Suppose that $n_0 > 2$. Then TFAE: for $2 < p_0 < n_0$,
(i) (G_{p_0}) holds, i.e., $|||\nabla e^{t\Delta}|||_{p_0 \to p_0} \leq \frac{C}{\sqrt{t}}$.
(ii) (GLY_{p_0}) holds:

$$(GLY_{p_0}) \quad \int_X |\nabla_x h_t(x,y)|^{p_0} \exp\left\{\gamma d(x,y)^2/t\right\} d\mu(x) \leq \frac{C}{t^{p_0/2}[V(y, \sqrt{t})]^{p_0-1}}.$$

(iii) (RH_p^E) holds, where (RH_p^E) means that there exists $C > 0$ such that for any ball B, with $3B \cap M_0 = \emptyset$, and any harmonic function u on $2B$, it holds

$$(RH_p^E) \qquad \left(\fint_B |\nabla u|^p \, d\mu\right)^{1/p} \leq \frac{C}{r_B} \fint_{2B} |u| \, d\mu.$$

Riesz tranforms on manifolds M: Let Δ be the Laplace-Beltrami operator on M. We consider the Riesz transform $\nabla(-\Delta)^{-1/2}$ on M, where $(-\Delta)^{-1/2}$ is given as

$$(-\Delta)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{s\Delta} \frac{ds}{\sqrt{s}}.$$

One considers

$$\nabla(-\Delta)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{s\Delta} \frac{ds}{\sqrt{s}}.$$

We always have for $f \in C_c^\infty(M)$,

$$\|\nabla(-\Delta)^{-1/2} f\|_2 = \|f\|_2.$$

For $p > 2$, consider the following conditions:
(i) (RH_p) holds: for any ball B and any harmonic function u on $2B$,

$$(RH_p) \qquad \left(\fint_B |\nabla u|^p \, d\mu\right)^{1/p} \leq \frac{C}{r_B} \fint_{2B} |u| \, d\mu.$$

(ii) (GLY_p) holds:

$$(GLY_p) \qquad \int_X |\nabla_x h_t(x,y)|^p \exp\left\{\gamma d(x,y)^2/t\right\} d\mu(x) \leq \frac{C}{t^{p_0/2}[V(y, \sqrt{t})]^{p-1}}.$$

(iii) (G_p) holds:

$$(G_p) \qquad \||\nabla H_t|\|_{p\to p} \leq C/\sqrt{t}.$$

(iv) (R_p) holds: $\|\nabla(-\Delta)^{-1/2} f\|_p \leq C\|f\|_p$.

If (D) and (P_2) hold, equivalently (LY), then by Auscher-Coulhon-Duong-Hofmann 2004 and Coulhon-J.-Koskela-Sikora 2020, for any $2 < p < \infty$

$$(RH_p) \Leftrightarrow (GLY_p) \Leftrightarrow (G_p) \Leftrightarrow (R_p);$$

If (D), (UE) and (P_2^E) hold, then by J. 2021, for $2 < p < n_0$,

$$(RH_p) \Leftrightarrow (GLY_p) \Leftrightarrow (G_p) \Leftrightarrow (R_p);$$

If $M = M_1 \# \cdots M_\ell$, assuming each M_i satisfies (D), (UE) and (P_2^E), then we *expect* that $2 < p < n_0$,

$$(RH_p^E) \Leftrightarrow (GLY_p) \Leftrightarrow (G_p) \Leftrightarrow (R_p).$$

Theorem (Coulhon-Duong-Li, 2003)

On any complete manifold, the Littlewood-Paley function

$$g_\Delta(f)(x) = \left(\int_0^\infty |\nabla e^{t\Delta} f(x)|^2 \, dt \right)^{1/2}$$

is bounded on $L^p(M)$ for $1 < p < 2$.

Proof Set $u(x,t) := e^{t\Delta} f(x)$, where $0 \le f \in C_c^\infty(M)$. For $1 < q < 2$,

$$|\nabla u(y,t)|^2 = \frac{1}{q(q-1)} u(y,t)^{2-q} J(y,t),$$

where $J(y,t) = -(\partial_t + \Delta) u(y,t)^q$. Note that $\int_M \Delta u(y,t)^q = 0$ and $J(y,t) \ge 0$.

This implies that

$$g_\Delta(f)(x) \le \sup_{t>0} |u(x,t)|^{\frac{2-q}{2}} \left(\int_0^\infty \frac{1}{q(q-1)} J(x,t)\, dt \right)^{1/2},$$

and hence,

$$\begin{aligned}
\|g_\Delta(f)\|_q &\le C \left\| \sup_{t>0} |u(x,t)|^{\frac{(2-q)q}{2}} \right\|_{\frac{2}{2-q}}^{\frac{1}{q}} \left(\int_M \int_0^\infty J(x,t)\, dt\, d\mu \right)^{1/2} \\
&\le C \|f\|_q^{\frac{2-q}{2}} \left(\int_M \int_0^\infty -\partial_t u(x,t)^q \, dt\, d\mu \right)^{1/2} \\
&\le C \|f\|_q^{\frac{2-q}{2}} \|f\|_q^{q/2} \\
&\le C \|f\|_q.
\end{aligned}$$

Conjecture (Coulhon-Duong, 2003)

The Riesz transform $\nabla(-\Delta)^{-1/2}$ is bounded on $L^p(M)$ for $1 < p < 2$ on complete manifolds.

Theorem (J.-Li-Lin, 2022)

Let $2 \le \ell \in \mathbb{N}$. Suppose that $\{M_i\}_{i=1}^{\ell}$ are complete, connected and non-collapsed manifolds of the same dimension, and each M_i satisfies (D) and (UE). Moreover, assume that for each i, for some $x_i \in M_i$ and all $R \ge r \ge 1$, it holds either

$$c_i \left(\frac{R}{r}\right)^2 \le \frac{V_i(x_i,R)}{V_i(x_i,r)} \le C_i \left(\frac{R}{r}\right)^2,$$

or for some $n_i > 2$

$$c_i \left(\frac{R}{r}\right)^{n_i} \le \frac{V_i(x_i,R)}{V_i(x_i,r)}.$$

Let $M = M_1 \# M_2 \# \cdots \# M_\ell$. Then the Riesz transform $\nabla(-\Delta)^{-1/2}$ is bounded on $L^p(M)$ for each $1 < p < 2$.

Theorem (J.-Li-Li-Shen 2024)

Let $M = M_1 \# M_2 \# \cdots \# M_\ell$ $(2 \le \ell \in \mathbb{N})$ be a connected sum of complete, non-compact, connected and non-collapsed manifolds, and each M_i satisfies (LY) and (RCA).

Assume that for each M_i, one of the following two conditions holds:

(i) There exist constants $n_i > 2$ and $c_i > 0$ such that, for some $x_i \in M_i$ and any $1 \le r \le R < \infty$, $c_i \left(\frac{R}{r}\right)^{n_i} \le \frac{V_i(x_i,R)}{V_i(x_i,r)}$.

(ii) There exist constants $1 \le n_i \le 2$, $c_i > 0$ and $C_i > 0$ such that, for some $x_i \in M_i$ and any $1 \le r \le R < \infty$, $c_i \left(\frac{R}{r}\right)^{n_i} \le \frac{V_i(x_i,R)}{V_i(x_i,r)} \le C_i \left(\frac{R}{r}\right)^{n_i}$.

Suppose that there is some $1 \le i \le \ell$ such that $n_i = 2$, and it holds

$$1 \le \min\{n_i : 1 \le i \le \ell\} < 2 < \max\{n_i : 1 \le i \le \ell\}.$$

Then the Riesz transform $\nabla \Delta^{-1/2}$ is bounded on $L^q(M)$ for each $1 < q < 2$.

(RCA) condition: there exists $A > 1$ such that, for all $R > 0$ large enough and for any two points $x, y \in M$ both at distance R from o, there is a continuous path γ connecting x to y and staying in the annulus $B(o, AR) \setminus B(o, R/A)$.

Remark:

(i) We do not know how to prove weak $(1, 1)$ boundedness. Our method uses the mapping property of the operators $\nabla e^{t\Delta}$ and $t\Delta e^{t\Delta}$, which only have optimal bounds for $p > 1$.

(ii) The requirement of the case $n_i = 2$ is stronger than the case $n_i > 2$, since $n_i = 2$ corresponds to the critical case.

(iii) Since we only need a upper Gaussian bound of the heat kernel on M_i, our result applies to any uniformly elliptic operators on these manifolds.

4. Application to the Riesz transform

(iv) Our result can be applied to the case where the manifolds have volume growth different from Ahlfors growth, which seems to be also new. For $\alpha \in (0,2)$, consider $\mathcal{R}^\alpha := (\mathbb{R}^2, g_\alpha)$, where g_α is a Riemannian metric such that, in the polar coordinates (ρ,θ), for $\rho > 1$ it equals

$$g_\alpha = \, d\rho^2 + \rho^{2(\alpha-1)}\, d\theta^2.$$

The volume of balls $B(x,r)$ on $\mathcal{R}^\alpha$, $r > 1$, has growth as

$$V(x,r) \sim \begin{cases} r^\alpha, & |x| < r \\ \min\{r^2,\ r|x|^{\alpha-1}\}, & |x| \geq r. \end{cases}$$

In particular, $V(0,r) \sim r^\alpha$ for $r > 1$. Note that for $\alpha \in (0,2)$, the exterior part $\{\rho > 1\}$ of $\mathcal{R}^\alpha$ is isometric to a certain surface of revolution in $\mathbb{R}^3$ and the Li-Yau estimate holds on $\mathcal{R}^\alpha$. For $n \geq 4$, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be closed manifolds of dimension $n-4$ and $n-2$ respectively. Our result then applies to the gluing manifolds $\mathbb{R}^n \# (\mathbb{R}^2 \times \mathcal{R}^\alpha \times \mathcal{M}_1)$ and $(\mathbb{R}^2 \times \mathcal{R}^\alpha \times \mathcal{M}_1) \# \mathbb{R}^2 \times \mathcal{M}_2$.

Proposition

The operator $\nabla e^{t\Delta}$ *is bounded on* $L^p(M)$ *for* $1 < p \leq 2$ *with*

$$\|\nabla e^{t\Delta}\|_{p\to p} \lesssim_p \frac{1}{\sqrt{t}}, \qquad \forall t > 0.$$

Remark

The above proposition and the followings hold on any complete manifolds.

Proposition

The operator $\nabla e^{t\Delta}$ is bounded on $L^p(M)$ for $1 < p \le 2$ with

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Remark

The above proposition and the followings hold on any complete manifolds.

For $1 \leq p < \infty$, we say that an operator T satisfies the L^p-Davies-Gaffney estimate, if

$$\|T(f\chi_E)\|_{L^p(F)} \leq C \exp\left(-\frac{d(E,F)^2}{Ct}\right) \|f\|_{L^p(E)}.$$

When $p = 2$, we shall say that T satisfies the Davies-Gaffney estimate for short.

Proposition

The operators $e^{t\Delta}$, $\sqrt{t}\nabla e^{t\Delta}$ and $t\Delta e^{t\Delta}$ satisfy the Davies-Gaffney estimate.

Using the Riesz-Thorin interpolation theorem, we further deduce that

Corollary

The operators $e^{t\Delta}$, $\sqrt{t}\nabla e^{t\Delta}$ and $t\Delta e^{t\Delta}$ satisfy L^p-Davies-Gaffney estimate for $1 < p \leq 2$.

4. Application to the Riesz transform

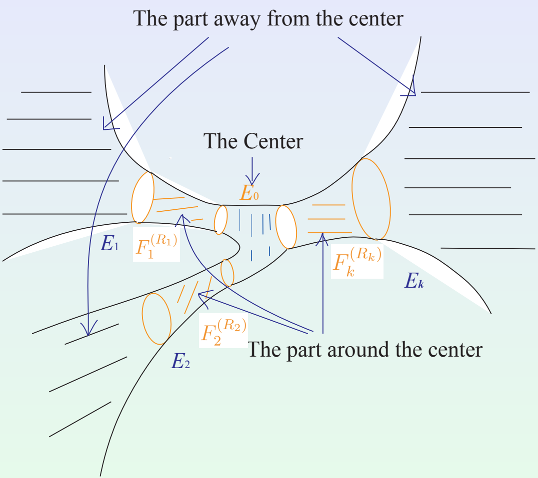

Figure: Decomposition of the manifold

Recall that for $r \geq 1$,

$$F_i^{(r)} := \{x \in E_i : \ \mathrm{dist}\,(x, E_0) \leq 2r\}, \qquad 1 \leq i \leq \ell.$$

If λ is large enough, saying $\lambda \geq 100\ell$, we set in the sequels,

$$R_i := R_i(\lambda) > 1 \text{ such that } \mu\left(F_i^{(R_i)}\right) = \lambda.$$

It holds then that

$$\mu(F_i^{(R_i)}) \sim V_i(R_i) \sim V_j(R_j) \sim \mu(F_j^{(R_j)}), \qquad \forall \lambda \gg 1,\ 1 \leq i, j \leq \ell.$$

Proposition

Under the assumptions of Theorem, for each $p \in [1,2)$, it holds that:

$$\left\| e^{t\Delta} \right\|_{L^p(F_i^{(R_i)}) \to L^2(E_j \setminus F_j^{(R_j)})} \lesssim \frac{R_j}{\sqrt{t}} \mu(F_i^{(R_i)})^{\frac{1}{2}-\frac{1}{p}}, \qquad \forall t \geq R_j^2,\ \lambda \gg 1,$$

and

$$\left\| e^{t\Delta} \right\|_{L^p(F_i^{(R_i)}) \to L^\infty(E_j \setminus F_j^{(R_j)})} \lesssim \frac{R_j^2}{t} \mu(F_i^{(R_i)})^{-\frac{1}{p}}, \qquad \forall t \geq R_j^2,\ \lambda \gg 1.$$

Hence, we obtain that

$$\left\|e^{t\Delta}\right\|_{L^1(F_i^{(R_i)})\to L^\infty(E_j\setminus F_j^{(R_j)})} \lesssim \frac{1}{t}\,\frac{R_j^2}{V_j(R_j)}, \qquad \forall t\ge R_j^2,\ \lambda\gg 1.$$

Next, recall that $\|e^{-t\Delta}\|_{L^1\to L^1}\le 1$. Hence, the Riesz-Thorin interpolation theorem implies that

$$\left\|e^{t\Delta}\right\|_{L^1(F_i^{(R_i)})\to L^2(E_j\setminus F_j^{(R_j)})} \lesssim \frac{1}{\sqrt{t}}\,\sqrt{\frac{R_j^2}{V_j(R_j)}}, \qquad \forall t\ge R_j^2,\ \lambda\gg 1.$$

By the fact that $\mu(F_i^{(R_i)}) \sim V_i(R_i) \sim V_j(R_j)$, then it follows from the Hölder inequality that

$$\left\| e^{t\Delta} \right\|_{L^p(F_i^{(R_i)}) \to L^\infty(E_j \setminus F_j^{(R_j)})} \lesssim \frac{1}{t} \frac{R_j^2}{V_j(R_j)} \mu(F_i^{(R_i)})^{1-\frac{1}{p}} \sim \frac{R_j^2}{t} \mu(F_i^{(R_i)})^{-\frac{1}{p}},$$

and

$$\left\| e^{t\Delta} \right\|_{L^p(F_i^{(R_i)}) \to L^2(E_j \setminus F_j^{(R_j)})} \lesssim \frac{1}{\sqrt{t}} \sqrt{\frac{R_j^2}{V_j(R_j)}} \mu(F_i^{(R_i)})^{1-\frac{1}{p}} \sim \sqrt{\frac{R_j^2}{t}} \mu(F_i^{(R_i)})^{\frac{1}{2}-\frac{1}{p}}.$$

This completes the proof.

We have the following main estimate

Proposition

Under the assumptions of the main Theorem, for each $p \in (1,2)$, it holds that

$$\|t\Delta e^{t\Delta}\|_{L^p(F_i^{(R_i)}) \to L^2(E_j \setminus F_j^{(R_j)})} \lesssim_p \mu(F_i^{(R_i)})^{\frac{1}{2}-\frac{1}{p}}, \qquad \forall t \geq R_j^2,\ \lambda \gg 1.$$

Lemma

Let $\epsilon \in (0, 1/8)$. Then:
(i) We have that

$$\left| t\,\partial_t h_t(x,y) \right| \lesssim_\epsilon \frac{1}{t}\frac{s^2}{V_i(s)}\left(1+\left(\frac{V_i(s)}{s^2}\right)^{\frac{3}{2}\epsilon}\left(\frac{|y|^2}{V_i(|y|)}\right)^{\frac{3}{2}\epsilon}\right)$$

$$\lesssim_\epsilon V_i(s)^{\frac{3}{2}\epsilon-1}\left(\frac{1}{V_i(|y|)}\right)^{\frac{3}{2}\epsilon}, \forall t \geq s^2 \geq 1, x \in E_i \setminus F_i^{(s)}, y \in F_i^{(s)}.$$

(ii) It holds that for all $1 \leq i \neq j \leq \ell$, $r, s \geq 1$,

$$\left| t\,\partial_t h_t(x,y) \right| \lesssim_\epsilon V_j(s)^{\frac{3}{2}\epsilon-1}\left(\frac{1}{V_i(|y|)}\right)^{\frac{3}{2}\epsilon}, \qquad \forall t \geq s^2, x \in E_j \setminus F_j^{(s)}, y \in F_i^{(r)}.$$

It follows from previous Lemma that

$$\left\| t\Delta e^{t\Delta}(f\chi_{F_i^{(R_i)}}) \right\|_{L^\infty(E_j\setminus F_j^{(R_j)})} \lesssim_\epsilon V_j(R_j)^{\frac{3}{2}\epsilon-1} \int_{F_i^{(R_i)}} |f(y)| \left(\frac{1}{V_i(|y|)}\right)^{\frac{3}{2}\epsilon} d\mu(y).$$

Then by the Hölder inequality, we obtain that

$$\begin{aligned} \left\| t\Delta e^{t\Delta}(f\chi_{F_i^{(R_i)}}) \right\|_{L^\infty(E_i\setminus F_i^{(R_i)})} &\lesssim_\epsilon V_j(R_j)^{\frac{3}{2}\epsilon-1} V_i(R_i)^{\frac{1}{p'}-\frac{3}{2}\epsilon} \|f\|_{L^p(F_i^{(R_i)})} \\ &\lesssim_\epsilon \frac{\|f\|_{L^p(F_i^{(R_i)})}}{\mu(F_i^{(R_i)})^{1/p}}, \end{aligned}$$

since $\mu(F_i^{(R_i)}) \sim V_i(R_i) \sim V_j(R_j)$.

In conclusion, for any $1 < p < 2$, by suitably choosing ϵ, we have that

$$\left\|t\Delta e^{t\Delta}\right\|_{L^p(F_i^{(R_i)})\to L^\infty(E_j\setminus F_j^{(R_j)})} \lesssim_p \frac{1}{\mu(F_i^{(R_i)})^{1/p}}.$$

On the other hand, the classical Littlewood-Paley-Stein theory says that

$$\left\|t\Delta e^{t\Delta}\right\|_{L^p(F_i^{(R_i)})\to L^p(E_j\setminus F_j^{(R_j)})} \lesssim_p 1, \qquad \forall\, 1 < p < +\infty.$$

The Hölder inequality implies that

$$\left\|t\Delta e^{t\Delta}\right\|_{L^p(F_i^{(R_i)})\to L^2(E_j\setminus F_j^{(R_j)})} \lesssim_p \mu(F_i^{(R_i)})^{\frac{1}{2}-\frac{1}{p}},$$

The keys in the proof include:

(i) The theory of Coulhon-Duong;

(ii) Grigoryan-Saloff-Coste's theory on heat kernels;

(iii) mapping property of $e^{t\Delta}$ and $\partial e^{t\Delta}$, including L^p-Davies-Gaffney estimates, $L^p(F_i) \to L^q(E_i\backslash)$ type estimates; (where the method of Davies plays a key role).

Thank you!