Sharp criteria for nonlocal elliptic inequalities on manifolds

Yuhua Sun (Nankai University)

Workshop on Heat Kernel and Related Topics in Hangzhou 2024

Joint work with

- Qingsong Gu (Nanjing University)
- Xueping Huang (Nanjing University of Information Science & Technology)

Q. Gu, X. Huang, Y. Sun, Sharp criteria for nonlocal elliptic inequalities on manifolds, arXiv: 2304.02808, preprint.

Let M be a noncompact geodesically complete manifold. We are interested in the existence and nonexistence of positive solutions to

$$(-\Delta)^{\alpha} u \geq u^{q} \sigma, \tag{$\diamond$}$$

where $\alpha \in (0, 1)$, and σ is a Radon measure on M.

Denote $d(x, y)$ by the geodesic distance on M, and μ by the Riemannian measure on M. Set

$$B(x, r) = \{y \in M : d(x, y) < r\}.$$

Let o be a reference point on M. Denote $B_r = B(o, r)$ and

$$V(r) = \mu(B(o, r)).$$

The following two conditions are very important in geometric analysis.

1. The manifold (M, g) is said to satisfy the volume doubling condition if for all $x \in M$ and $r > 0$,

$$\mu(B(x, 2r)) \lesssim \mu(B(x, r)); \tag{VD}$$

2. The manifold (M, g) is said to satisfy the Poincaré inequality if for any ball $B = B(x, r) \subset M$ and any $f \in C^1(B)$,

$$\int_B |f - f_B|^2 d\mu \lesssim r^2 \int_B |\nabla f|^2 d\mu, \tag{PI}$$

where $f_B = \frac{1}{\mu(B)} \int_B f d\mu$.

Various points of view

1. Integral operator: in $\mathbb{R}^n$,

$$(-\Delta)^\alpha u(x) = C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\alpha}} dx;$$

2. Fourier method: $\mathbb{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha} \mathfrak{F}(u)$;
3. Extension method: Molchanov and Ostrovskii ('69), Caffarelli and Silvestre ('07);
4. Subordination(functional analysis): $-\Delta$ is non-negative self-adjoint, and $\lambda \mapsto \lambda^\alpha$ is a Bernstein function;
5. Subordination(stochastic process): time change of the Brownian motion $(B_t)_{t\geq 0}$ by an independent subordinator $(S_t)_{t\geq 0}$: $X_t = B_{S_t}$.

Subordination method

Consider the spectral representation of $-\Delta$:

$$-\Delta = \int_0^\infty \lambda dE_\lambda,$$

where $\{E_\lambda\}_{\lambda\geq 0}$ is the projection operator valued measure associated to $-\Delta$. The heat semigroup generated by Δ, then $P_t = e^{t\Delta}$ can be represented as

$$P_t = \int_0^\infty e^{-t\lambda} dE_\lambda.$$

The heat semigroup admits positive smooth densities $\{p_t(x,y)\}_{t>0}$, namely

$$P_t f(x) = \int_M p_t(x,y) f(y) d\mu(y), \quad \forall t > 0, x \in M,$$

which is valid for $f \in L^2(M,\mu)$.

The Green operator corresponding to $-\Delta$ is defined by

$$G = \int_0^\infty P_t dt,$$

with integral density

$$G(x,y) = \int_0^\infty p_t(x,y) dt,$$

which is possibly infinite everywhere.

Fix $\alpha \in (0,1)$, we can define the fractional Laplacian $(-\Delta)^\alpha$ as

$$(-\Delta)^\alpha = \int_0^\infty \lambda^\alpha dE_\lambda. \tag{0.1}$$

By subordination theory, $(-\Delta)^\alpha$ is self-adjoint and generates another semigroup, which we denote by $(P_t^{(\alpha)})_{t\geq 0}$.

and the new semigroup $(P_t^{(\alpha)})_{t\geq 0}$ can then be represented in terms of the original one as

$$P_t^{(\alpha)} = \exp\left(-t(-\Delta)^\alpha\right) = \int_0^\infty \exp\left(-t\lambda^\alpha\right) dE_\lambda$$

Definition

The Green kernel $G^{(\alpha)}(\cdot,\cdot)$ associated with the fractional Laplacian $(-\Delta)^\alpha$ is defined as

$$G^{(\alpha)}(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} p_s(x,y) ds, \quad x, y \in M. \tag{0.2}$$

We say that the fractional Laplacian $(-\Delta)^\alpha$ is transient if there exists a non-negative measurable function f on M with $\mu\left(\{x \in M : f(x) > 0\}\right) > 0$ such that

$$\int_M G^{(\alpha)}(x,y) f(y) d\mu(y) < +\infty, \tag{0.3}$$

for μ-a.e. $x \in M$.

For a complete connected non-compact Riemannian manifold (M, g), the classical works of Grigor'yan, and Saloff-Coste show that the combination of (VD) and (PI) is equivalent to

$$\frac{1}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2(x,y)}{c_2 t}} \lesssim p_t(x, y) \lesssim \frac{1}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2(x,y)}{c_1 t}}.$$

Proposition (G^α estimate)

Let (M, g) be a complete connected non-compact Riemannian manifold satisfying conditions (VD) and (PI). Then the Green kernel for $(-\Delta)^\alpha$ satisfies

$$G^{(\alpha)}(x, y) \asymp \int_{d(x,y)}^{+\infty} \frac{t^{2\alpha-1} dt}{\mu(B(x, t))}, \quad x, y \in M. \tag{0.4}$$

Moreover, the fractional Laplacian $(-\Delta)^\alpha$ is transient if and only if for some $x_0 \in M$,

$$\int_1^{+\infty} \frac{t^{2\alpha-1} dt}{\mu(B(x_0, t))} < \infty. \tag{0.5}$$

$\alpha-$**transience**: Assume that the operator $(-\Delta)^\alpha$ is transient, namely there is an associated Green kernel function $G^{(\alpha)}(\cdot,\cdot) : M \times M \to (0,+\infty]$ which is lower semi-continuous, finite off the diagonal, and is formally the inverse of $(-\Delta)^\alpha$

We know the positive solution to $(\diamond)$ is closely related to the positive solution to

$$u(x) \geq \int_M G^{(\alpha)}(x,y)u^q(y)d\sigma(y) \tag{$\spadesuit$}$$

Definition (Solution to Integral inequality)

Let u be a σ-a.e. defined function that admits a lower semi-continuous σ-version. Then u is called a positive solution to $(\spadesuit)$ if for σ-a.e. $x \in M$, $u(x) \in (0,+\infty)$, and $(\spadesuit)$ holds.

Main results

Theorem 1(Gu-Huang-S., 2023)

Let (M, g) be a complete connected non-compact Riemannian manifold satisfying conditions (VD) and (PI). Assume that for some $x_0 \in M$,

$$\int_1^{+\infty} \frac{t^{2\alpha-1} dt}{\mu(B(x_0, t))} < \infty.$$

Then there exists a positive solution to (♠) if and only if there exist $o \in M$, $r_0 > 0$ such that the following two conditions hold:

$$\int_{r_0}^{+\infty} \left[\int_r^{+\infty} \frac{t^{2\alpha-1} dt}{\mu(B(o,t))} \right]^{q-1} \frac{\sigma(B(o,r))}{\mu(B(o,r))} r^{2\alpha-1} dr < \infty,$$

and

$$\sup_{x \in M, r > r_0} \left[\int_0^{+\infty} \frac{\sigma(B(x,s) \cap B(o,r))}{\mu(B(x,s))} s^{2\alpha-1} ds \right] \left[\int_r^{+\infty} \frac{t^{2\alpha-1} dt}{\mu(B(o,t))} \right]^{q-1} < +\infty.$$

Theorem 2(Gu-Huang-S., 2023)

Let (M, g) be a complete connected non-compact Riemannian manifold satisfying conditions (VD) and (PI). Assume that for some $x_0 \in M$,

$$\int_1^{+\infty} \frac{t^{2\alpha-1}dt}{\mu(B(x_0, t))} < \infty.$$

Then there exists a lower semi-continuous function $u : M \to (0, +\infty)$ such that

$$u(x) \geq \int_M G^{(\alpha)}(x, y)u^q(y)d\mu(y),$$

for all $x \in M$, if and only if there exist $o \in M$ and $r_0 > 0$ such that

$$\int_{r_0}^{+\infty} \frac{r^{2\alpha q-1}}{[\mu(B(o, r))]^{q-1}}dr < \infty.$$

Quasi-metric property: there exists a constant $\kappa \geq 1$ such that

$$G^{(\alpha)}(x,y) \wedge G^{(\alpha)}(y,z) \leq \kappa G^{(\alpha)}(x,z), \quad \text{for all } x,y,z \in M.$$

and for $a > 0$ set

$$m(x) = m_{a,o}(x) = G^{(\alpha)}(x,o) \wedge a^{-1}.$$

Theorem 3(Gu-Huang-S., 2023)

Let M be a complete connected non-compact Riemannian manifold. Assume that the fractional Laplacian $(-\Delta)^{\alpha}$ is transient, and suppose that $G^{(\alpha)}$ is quasi-metric. Then the followings are equivalent:

1. There exists a positive solution to (♠) in the sense of Definition.
2. There exists a lower semi-continuous function $u : M \to (0,+\infty)$ such that (♠) holds for all $x \in M$.
3. For σ-a.e. $x \in M$,
$$m(x) \gtrsim \int_M G^{(\alpha)}(x,y)m^q(y)d\sigma(y).$$
4. For all $x \in M$,
$$m(x) \gtrsim \int_M G^{(\alpha)}(x,y)m^q(y)d\sigma(y).$$

Theorem 4(Gu-Huang-S., 2023)

Let M be a complete connected non-compact Riemannian manifold. Assume that the fractional Laplacian $(-\Delta)^{\alpha}$ is transient. Suppose that $G^{(\alpha)}$ is quasi-metric. Then there exists a positive solution to (♠) if and only if the following two conditions hold:

$$\int_M m^q(x) d\sigma(x) < \infty, \tag{0.6}$$

and

$$\sup_{x \in M} \int_{\{y \in M : G^{(\alpha)}(o,y) > r^{-1}\}} G^{(\alpha)}(x,y) d\sigma(y) \lesssim r^{q-1}, \tag{0.7}$$

for all $r > a$.

Definition (Solution to Differential inequality)

Let M be a complete connected non-compact Riemannian manifold. Assume that the fractional Laplacian $(-\Delta)^\alpha$ is transient and that σ is *absolutely continuous* with respect to the Riemannian measure μ. Denote by $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ the Dirichlet form associated with $(-\Delta)^\alpha$. Let $\mathcal{F}_e^{(\alpha)}$ be the corresponding extended Dirichlet space. A non-negative function $v \in \mathcal{F}_e^{(\alpha)}$ is said to be a positive solution to $(\diamond)$ in $\mathcal{F}_e^{(\alpha)}$-sense if $v > 0$ σ-a.e., and

$$\mathcal{E}^{(\alpha)}(v, \varphi) \geq \int_M v^q \varphi d\sigma,$$

for each $\varphi \in \mathcal{F}^{(\alpha)} \cap C_c(M)$ with $\varphi \geq 0$.

Theorem 5(Gu-Huang-S., 2023)

Let M be a complete connected non-compact Riemannian manifold. Assume that σ is absolutely continuous with respect to μ, with a continuous density function θ. Assume that the fractional Laplacian $(-\Delta)^{\alpha}$ is transient, and suppose that $G^{(\alpha)}$ is quasi-metric. Then there exists a positive solution to (♠) if and only if there is a positive solution to (♢) in $\mathcal{F}_e^{(\alpha)}$-sense.

Theorem 6(Gu-Huang-S., 2023)

Let M be a complete connected non-compact Riemannian manifold. Assume that the fractional Laplacian $(-\Delta)^\alpha$ is transient and that the Green kernel $G^{(\alpha)}$ is quasi-metric. Assume further that the heat semigroup $\{P_t\}_{t\geq 0}$ associated with Δ is Feller, and that

$$\lim_{d(x,o)\to+\infty} G^{(\alpha)}(x,o) = 0,$$

for some fixed reference point $o \in M$. Then there exists a positive solution to (♠), if and only if there is a positive smooth function h on M that is in the domain of $(-\Delta)^\alpha$ (on $C_\infty(M)$) and satisfies

$$(-\Delta)^\alpha h \geq h^q.$$

From Euclidean Space

It is well-known that for the equation

$$\Delta u + u^q = 0 \quad \text{in } \mathbb{R}^n$$

has a positive solution iff $n > 2$, and $q \geq \frac{n+2}{n-2}$. (Obata, Caffarelli-Gidas-Spruck, Gidas-Spruck, Chen-Li,...)

For the inequality

$$\Delta u + u^q \leq 0 \quad \text{in } \mathbb{R}^n$$

has a positive solution iff $n > 2$, and $q > \frac{n}{n-2}$. (Gidas-Spruck, Mitidieri, Pohozaev, Serrin...)

Theorem (Xiao-Wang, 2016)

There exists a positive solution to

$$(-\Delta)^{\alpha} u \geq u^{q}, \quad \text{in } \mathbb{R}^n,$$

iff $n > 2\alpha$, and $q > \frac{n}{n-2\alpha}$.

Remark. Using extension method+test function technique...

J. Xiao, Y. Wang, *A uniqueness principle for $u^p \leq (-\Delta)^{\frac{\alpha}{2}} u$ in the Euclidean Space*, Comm. Cont. Math., Vol.18, No. **6**(2016), 1650019.

From Manifolds

Though the volume of geodesic ball is a very simple geometric quantity of manifold, but it plays a very important role when one studies the geometric and probabilistic properties.

Volume growth can be used to give the information of the following

- parabolicity of manifold
- stochastic completeness of manifold
- heat kernel
- existence and nonexistence of partial differential inequalities and equations
- escape rate...

A. Grigor'yan, *Analysis on manifolds and volume growth*, Advances in Analysis and Geometry **3**(2021), de Gruyter, 299-324.

We say a manifold M is parabolic if any positive superharmonic function on M is constant.

Theorem (Cheng-Yau, 1975)

If there exists a sequence $r_k \to \infty$ such that for all $k > 0$

$$V(r_k) \lesssim r_k^2,$$

then M is parabolic.

Remark: M is parabolic$\Leftrightarrow$ any positve superharmonic function on M is constant $\Leftrightarrow$ Δ has no positive fundamental solution$\Leftrightarrow$ Brownian motion on M is recurrent.

S. Y. Cheng, S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28**(1975), 333-354.

Theorem (Karp 1982, Grigor'yan 1983, Varopoulos 1983)

If

$$\int^{\infty} \frac{r}{V(r)} dr = \infty, \tag{vol-pra}$$

then M is parabolic.

Remark: (vol-pra) is sharp in the sense that if there exists a smooth convex function $f(r)$ with $f'(r) > 0$, and

$$\int^{\infty} \frac{r}{f(r)} dr < \infty$$

then there exists a non-parabolic manifold with $V(r) = f(r)$ for large r. However, (vol-pra) is not a necessary condition for parabolicity of manifolds. (cf. Grigor'yan, [Bull. Amer. Math. Soc. 1999])

Theorem (Holopainen, 1999)

If

$$\int^{\infty} \left(\frac{r}{V(r)} \right)^{\frac{1}{m-1}} dr = \infty, \qquad \text{(vol-m-pra)}$$

then any nonnegative solution to $\Delta_m u \leq 0$ is identical constant.

Holopainen, Volume growth, Green's function, and parabolicity of Ends, Duke Math. J., vol.**97** (1999), no. 2,

Theorem (Grigor'yan-S., 2014)

If

$$V(r) \lesssim r^{Q_1}(\ln r)^{Q_2}, \qquad \text{(vol-sem)}$$

holds for all large enough r, then any non-negative solution to $\Delta u + u^p \leq 0$ with $p > 1$ is identically equal to zero. Here $Q_1 = \frac{2p}{p-1}, Q_2 = \frac{1}{p-1}$.

Remark: here the volume condition (vol-sem) is sharp, if the above exponents Q_1, Q_2 are relaxed, there exists some manifold which admits positive solution to $\Delta u + u^p \leq 0$.

A. Grigor'yan, Y. Sun, *On non-negative solution of the inequality $\Delta u + u^\sigma \leq 0$ on Riemannian manifolds*, Comm. Pure Appl. Math. **67** (2014) no. 8, 1336–1352.

A complete classification using the volume growth was obtained for

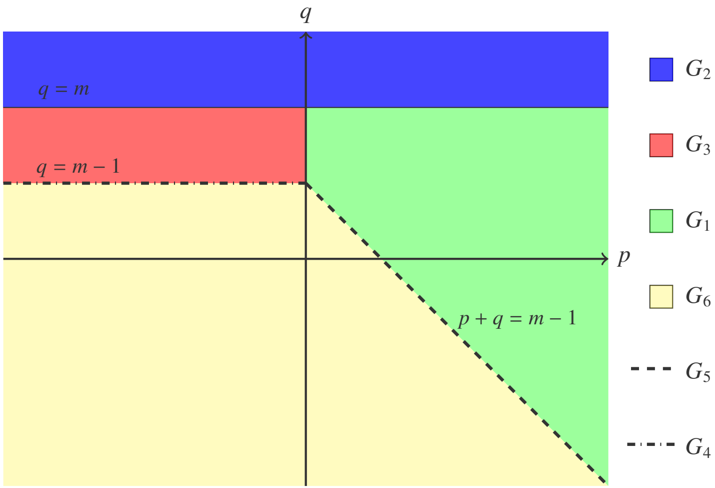
$$\Delta_m u + u^p |\nabla u|^q \leq 0, \quad \text{on } M, \tag{EI}$$

where $m > 1$, $\Delta_m u = \mathrm{div}(|\nabla u|^{m-2}\nabla u)$, and $(p, q) \in \mathbb{R}^2$ which means that p, q can be allowed to be negative.

Y. Sun, J. Xiao, F. Xu, *A sharp Liouville principle for* $\Delta_m u + u^p|\nabla u|^q \leq 0$ *on geodesically complete noncompact Riemannian manifolds*, Math. Ann. 384 (2022), no. 384, 1309-1341.

Let us divide $\mathbb{R}^2$ into six parts

$$G_1 = \{(p,q)|p \geq 0, m-1-p < q < m\}, \quad G_2 = \{(p,q)|q \geq m\},$$
$$G_3 = \{(p,q)|p < 0, m-1 < q < m\}, \quad G_4 = \{(p,q)|p < 0, q = m-1\},$$
$$G_5 = \{(p,q)|p = m-1-q, q \leq m-1\}, \quad G_6 = \{(p,q)|p < m-1-q, q < m-1\}.$$



Theorem(S.-Xiao-Xu, 2022)

Let M be a noncompact geodesically complete manifold.

(I) Assume $(p, q) \in G_1$. If

$$V(r) \lesssim r^{\frac{mp+q}{p+q-m+1}} (\ln r)^{\frac{m-1}{p+q-m+1}}, \quad \text{for all large enough } r,$$

then (**EI**) possesses no nontrivial positive solution.

(II) Assume $(p, q) \in G_2$. If

$$V(r) \lesssim r^m (\ln r)^{m-1}, \quad \text{for all large enough } r,$$

then (**EI**) possesses no nontrivial positive solution.

Theorem(S.-Xiao-Xu, 2022)

(III) Assume $(p,q) \in G_3$. If

$$V(r) \lesssim r^{\frac{q}{q-m+1}} (\ln r)^{\frac{m-1}{q-m+1}}, \quad \text{for all large enough } r,$$

then (**EI**) possesses no nontrivial positive solution.

(IV) Assume $(p,q) \in G_4$. Given $\alpha > 0$, if

$$V(r) \lesssim r^{\alpha}, \quad \text{for all large enough } r,$$

then (**EI**) possesses no nontrivial positive solution.

Theorem(S.-Xiao-Xu, 2022)

(V) Assume $(p,q) \in G_5$. Given $0 < \kappa < \frac{\min\{m-1,1\}}{2e}$, if

$$V(r) \lesssim e^{\kappa r}, \quad \text{for all large enough } r,$$

then (**EI**) possesses no nontrivial positive solution.

(VI) Assume $(p,q) \in G_6$. Given $0 < \kappa < \frac{m-1-q}{m-1-p-q}$, if

$$V(r) \lesssim e^{\kappa r \ln r}, \quad \text{for all large enough } r,$$

then (**EI**) possesses no nontrivial positive solution.

Integral form

Theorem (Grigor'yan-S.- Verbitsky, 2020)

Assume (VD) and (PI) are both satisfied on M. Then $\Delta u + u^p \leq 0$ admits a positive solution if and only if

$$\int^{\infty} \frac{r^{2p-1}}{V(r)^{p-1}} dr < \infty, \qquad \text{(vol-int-1)}$$

or equivalently

$$\int^{\infty} \left[\int_r^{\infty} \frac{t dt}{V(t)} \right]^{p-1} r dr < \infty, \qquad \text{(vol-int-2)}$$

holds.

A. Grigor'yan, Y. Sun, I. Verbitsky, *Superlinear elliptic inequalities on manifolds*, J. Funct. Anal. **278** (2020), no. 9, 108444, 34 pp.

It is easy to see that if for a given $\alpha > 2$, and

$$V(r) \lesssim r^{\alpha} \ln^{\frac{\alpha-2}{2}} r \cdots (\ln \cdots \ln r)^{\frac{\alpha-2}{2}},$$

then for $p \leq \frac{\alpha}{\alpha-2}$, $\Delta u + u^p \leq 0$ admits no positive solution.

Conjecture

If (vol-int-1)(or (vol-int-2)) is not satisfied on M, then any nonnegative solution to $\Delta u + u^p \leq 0$ is identical zero.

Comparsion of two methods

- Using test function only need less restriction on manifolds, but can only give nonexistence results, and at the same time cannot give you too sharp volume growth condition, for example, there are still "$\ln\ln$" room for self-improvement.
- Using potential theoretical approach can give you sharp criteria for existence and nonexistence via Green function.

Recall potential theoretical tools

1. (X, d): a locally compact, separable metric space;
2. $\mathcal{M}^+(X)$: the space of Radon measures on X which is the class of locally finite Borel measures on X.
3. a kernel on X: a Borel measurable function $K(\cdot, \cdot) : X \times X \to [0, +\infty]$.

For $\nu \in \mathcal{M}^+(X)$ and a nonnegative Borel measurable function f, we denote $K_\nu f(\cdot) = \int_X K(\cdot, y) f(y) d\nu(y)$. For example, $K_\nu 1(\cdot) = \int_X K(\cdot, y) d\nu(y)$.

We say that $\nu \in \mathcal{M}^+(X)$ is concentrated on A if for any non-negative measurable function g on X, $\int_X g d\nu = \int_A g d\nu$.

Possible properties of a kernel

A kernel K possibly satisfies:

1. K is positive if $K(x, y) \in (0, +\infty]$ for each $x, y \in X$;
2. K is symmetric if $K(x, y) = K(y, x)$ for each $x, y \in X$;
3. K is lower semi-continuous;
4. K is locally integrable with respect to $\nu \in \mathcal{M}^+(X)$ if $K(x, \cdot) \in L^1_{\text{loc}}(X, \nu)$ for each $x \in X$. (σ-finiteness)
5. $K(x, \cdot)$ is locally finite with respect to $\nu \in \mathcal{M}^+(X)$ if $K(x, \cdot) \in L^1_{\text{loc}}(X, \nu)$. Note that in our setting this implies that $K(x, \cdot)$ is σ-finite with respect to ν.
6. K is σ-finite (locally finite) with respect to ν if $K(x, \cdot)$ is σ-finite (locally finite) with respect to ν for each $x \in X$.

The weak maximum principle

Definition (The weak maximum principle)

A kernel K on a locally compact separable metric space (X, d) is said to satisfy the weak maximum principle with constant $b \geq 1$, if for each $\nu \in \mathcal{M}^+(X)$ and each Borel set $A \subset X$ such that ν is concentrated on A, we have

$$K_\nu 1 \leq 1 \quad \text{in } A \quad \Rightarrow \quad K_\nu 1 \leq b \quad \text{in } X. \tag{0.8}$$

Lemma 1

Let $\{K_n\}_{n\geq 1}$ be an increasing sequence of kernels on a locally compact separable metric space. Suppose that each of them satisfies the weak maximum principle with a common constant $b \geq 1$. Set

$$K = \lim_{n\to+\infty} K_n.$$

Then K satisfies the weak maximum principle with constant $b \geq 1$.

A convenient version of the weak maximum principle

Lemma 2

Let K be a kernel on (X, d) satisfying the weak maximum principle with constant $b \geq 1$. Suppose η is a non-negative measurable function. Define $\tilde{K}$ by $\tilde{K}(x, y) = K(x, y)\eta(y)$. Then $\tilde{K}$ satisfies the weak maximum principle with constant b as well.

Corollary 3

Let K be a kernel on (X, d) satisfying the weak maximum principle with constant $b \geq 1$. Suppose that f is a nonnegative measurable function. Let $\nu \in \mathcal{M}^+(X)$ be concentrated on a Borel set $A \subset X$. Then for $\nu \in \mathcal{M}^+(X)$,

$$K_\nu f \leq 1 \quad \text{in } \{f > 0\} \cap A \quad \Rightarrow \quad K_\nu f \leq b \quad \text{in } X.$$

The quasi-metric property

Definition (Quasi-metric)

A kernel K on X is quasi-metric, if K is symmetric and there exists a constant $\kappa \geq 1$ such that

$$K(x,y) \wedge K(y,z) \leq \kappa K(x,z), \quad \text{for all } x,y,z \in X. \tag{qm}$$

Lemma 4 (Frazier, Nazanov, Verbitsky, 2014)

Let K be a quasi-metric kernel on X with constant $\kappa \geq 1$. Then

$$(K(x,y)K(o,z)) \wedge (K(y,z)K(o,x)) \leq \kappa^2 (K(x,z)K(o,y)).$$

Lemma 5 (Quinn, Verbitsky, 2018)

If K is a quasi-metric kernel on (X,d) with constant κ, then K satisfies the weak maximum principle with constant $b = \kappa$.

The quasi-metric property

Lemma 6 (Quinn, Verbitsky, 2018)

Let K be a positive, quasi-metric kernel with constant $\kappa \geq 1$. Fix $o \in X$ and $c > 0$. Let $k(\cdot) = K(o, \cdot) \wedge c$. Let $\tilde{K} : X \times X \to (0, +\infty]$ defined by

$$\tilde{K}(x, y) = \frac{K(x, y)}{k(x)k(y)}, \quad x, y \in X.$$

Then for all $x, y, z \in X$,

$$\tilde{K}(x, y) \wedge \tilde{K}(y, z) \leq \kappa^2 \tilde{K}(x, z).$$

In particular, $\tilde{K}$ satisfies the weak maximum principle with constant κ^2.

M. Frazier, F. Nazarov, and I. Verbitsky, Global estimates for kernels of Neumann series and Green's functions, J. London Math. Soc. **90** (3) (2014), 903-918.

S. Quinn, I. E. Verbitsky, *A sublinear version of Schur's lemma and elliptic PDE*, Anal. PDE **11** (2018), no. 2, 439-466.

Some Useful Estimates

Lemma 7 (Grigor'yan, Verbitsky, 2020)

Let (Ω, ω) be a σ-finite measure space, and let $0 < a = \omega(\Omega) \leq +\infty$. Let $f : \Omega \to [0, +\infty]$ be a measurable function. Let $\varphi : [0, a) \to [0, +\infty)$ be a continuous, monotone non-decreasing function, and set $\varphi(a) := \lim_{t \to a^-} \varphi(t) \in (0, \infty]$. Then the following inequality holds:

$$\int_0^{\omega(\Omega)} \varphi(t) dt \leq \int_\Omega \varphi\left(\omega(\{z \in \Omega : f(z) \leq f(y)\})\right) d\omega(y). \tag{0.9}$$

A. Grigor'yan, I. Verbitsky, *Pointwise estimates of solutions to nonlinear equations for nonlocal operators*, Ann. Scuola Norm. Sup. Pisa, XX (2020), 721-750.

Lemma 8 (Grigor'yan, Verbitsky, 2020)

Let (X, d) be a locally compact separable metric space. Let $\nu \in \mathcal{M}^+(X)$. Assume that K satisfies the weak maximum principle with constant $b \geq 1$. Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a continuous, non-decreasing function, and set

$$\varphi(+\infty) := \lim_{t \to +\infty} \varphi(t) \in [0, +\infty].$$

Then

$$\int_0^{K_\nu 1(x)} \varphi(t) dt \leq K_\nu[\varphi(bK_\nu 1)](x),$$

for each $x \in X$ such that $K(x, \cdot)$ is σ-finite with respect to ν.

Corollary 9 (Grigor'yan, Verbitsky, 2020)

Let $q > 1$. Under the same conditions as in above lemma, the following inequality holds for each $x \in X$ such that $K(x, \cdot)$ is σ-finite with respect to ν,

$$(K_\nu 1(x))^q \leq q b^{q-1} (K_\nu ((K_\nu 1)^{q-1})(x).$$

Lemma 10 (Grigor'yan, Verbitsky, 2020)

Under the setting of Lemma 7, define a sequence $\{f_k\}_{k\geq 0}$ of functions on X by

$$f_0 = K_\nu 1, \quad f_{k+1} = K_\nu(\varphi(f_k)) \quad \text{for } k \in \mathbb{N}. \tag{0.10}$$

Set for $t \geq 0$

$$\psi(t) = \varphi(b^{-1}t) \tag{0.11}$$

and define also the sequence $\{\psi_k\}_{k=0}^{\infty}$ of functions on $[0, \infty)$ by $\psi_0(t) = t$ and

$$\psi_{k+1}(t) = \int_0^t \psi \circ \psi_k(s) ds, \quad \text{for } k \in \mathbb{N}. \tag{0.12}$$

Then, for each $k \in \mathbb{N}$

$$\psi_k(f_0(x)) \leq f_k(x), \tag{0.13}$$

for each $x \in X$ such that $K(x, \cdot)$ is σ-finite with respect to ν.

Corollary 11 (Grigor'yan, Verbitsky, 2020)

Under the setting of Lemma 7, choose $\varphi(t) = t^q$ for some $q > 0$. We have for each $x \in X$ such that $K(x, \cdot)$ is σ-finite with respect to ν,

$$[K_\nu 1(x)]^{1+q+q^2+\cdots+q^k} \leq b^{q+q^2+\cdots+q^k} c(q,k) f_k(x), \tag{0.14}$$

where $c(q,k) = \prod_{j=1}^{k}(1 + q + q^2 + \cdots + q^j)^{q^{k-j}}$.

A refined Version

Theorem 12 (Kalton, Verbitsky, 1999, Grigor'yan, Verbitsky, 2020)

Let (X, d) be a locally compact separable metric space. Let $\nu \in \mathcal{M}^+(X)$. Let h be a bounded, positive measurable function on X. Assume that the new kernel $\tilde{K}$ defined by $\tilde{K}(x, y) = \frac{K(x,y)}{h(x)h(y)}$ satisfies the weak maximum principle with constant $b \geq 1$.

Let $q > 1$ be a constant. Suppose that $u : X \to (0, +\infty]$ is a Borel measurable function. Assume that there is a measurable set $A \subseteq X$ with $\nu(A^c) = 0$ such that for each $x \in A$, $u(x) < +\infty$, and

$$u(x) \geq K_\nu(u^q)(x) + h(x)$$

holds. Then at each $x \in A$, we have

$$K_\nu(h^q)(x) < \frac{b}{q-1} h(x).$$

N. J. Kalton and I. E. Verbitsky, *Nonlinear equations and weighted norm inequalities*, Trans. Amer. Math. Soc. **351** (1999) 3441-3497.

Lemma 13

Let K be a lower semi-continuous quasi-metric positive kernel on (X, d) with constant κ. Fix $o \in X$ and $a > 0$. Set $m(x) = K(o, x) \wedge a^{-1}$. Then for each positive measure ω of the form $\omega = f\nu$ with f non-negative Borel measurable and ν a Radon measure, we have for all $x \in X$,

$$K_\omega 1(x) \gtrsim m(x).$$

Quasi-metric property: there exists a constant $\kappa \geq 1$ such that

$$G^{(\alpha)}(x,y) \wedge G^{(\alpha)}(y,z) \leq \kappa G^{(\alpha)}(x,z), \quad \text{for all } x,y,z \in M.$$

and set

$$m(x) = m_{a,o}(x) = G^{(\alpha)}(x,o) \wedge a^{-1}.$$

Theorem 3(Gu-Huang-S., 2023)

Let M be a complete connected non-compact Riemannian manifold. Assume that the fractional Laplacian $(-\Delta)^{\alpha}$ is transient, and suppose that $G^{(\alpha)}$ is quasi-metric. Then the followings are equivalent:

1. There exists a positive solution to (♠) in the sense of Definition.
2. There exists a lower semi-continuous function $u : M \to (0,+\infty)$ such that (♠) holds for all $x \in M$.
3. For σ-a.e. $x \in M$,
$$m(x) \gtrsim \int_M G^{(\alpha)}(x,y) m^q(y) d\sigma(y).$$
4. For all $x \in M$,
$$m(x) \gtrsim \int_M G^{(\alpha)}(x,y) m^q(y) d\sigma(y).$$

Proof of Theorem 3

"(3) $\Longrightarrow$ (4)": Let A be the set of $x \in X$ such that

$$m(x) \gtrsim \int_M G^{(\alpha)}(x,y)m^q(y)d\sigma(y).$$

By assumption σ is concentrated on A. Since $G^{(\alpha)}(\cdot,\cdot)$ is quasi-metric, by Lemmas 2 and 6, the kernel $\hat{G}^{(\alpha)}(\cdot,\cdot)$ defined by

$$\hat{G}^{(\alpha)}(x,y) = \frac{1}{m(x)}G^{(\alpha)}(x,y)m(y)^q$$

satisfies the weak maximum principle. It follows that

$$m(x) \gtrsim \int_M G^{(\alpha)}(x,y)m^q(y)d\sigma(y),$$

for each $x \in X$, with a possibly different constant.

"(4) $\Longrightarrow$ (2)": This is clear since $m(x) = G^{(\alpha)}(x,o) \wedge a^{-1}$ is lower semi-continuous, and is positive and finite everywhere. The inequality (♠) holds for cm with some suitable constant $c > 0$.

"(2) $\Longrightarrow$ (1)": This is trivial.

"(1) $\Longrightarrow$ (3)": Suppose that u is a positive solution to (♠). Then $u : M \to [0, +\infty]$ is lower semi-continuous and we can find some measurable set $A \subseteq M$ with $\sigma(A^c) = 0$ such that $u(x) \in (0, +\infty)$ and

$$u(x) \geq \int_M G^{(\alpha)}(x, y) u^q(y) d\sigma(y),$$

for each $x \in A$.

Note that for each $x \in A$,

$$+\infty > u(x) \geq \int_M G^{(\alpha)}(x, y) u^q(y) d\sigma(y) = \int_A G^{(\alpha)}(x, y) u^q(y) d\sigma(y).$$

Hence $G^{(\alpha)}(x, \cdot)$ is σ-finite with respect to ν for each $x \in A$.

Fix some $\varepsilon \in (0, 1)$, let $v = \varepsilon u$. Then

$$v \geq G^{(\alpha)}_\sigma(v^q) + (\varepsilon^{1-q} - 1) G^{(\alpha)}_\sigma(v^q)$$

holds on A. Since v is lower semi-continuous and σ-a.e. positive, by Lemma 13, there is a constant $C > 0$ such that

$$(\varepsilon^{1-q} - 1) G^{(\alpha)}_\sigma(v^q) = (\varepsilon^{1-q} - 1) G^{(\alpha)}_{v^q\sigma} 1 \geq C(\varepsilon^{1-q} - 1) m.$$

Hence v satisfies the following inequality for each $x \in A$,

$$v(x) \geq G^{(\alpha)}_\sigma(v^q)(x) + C m(x).$$

By Lemma 6, the kernel $\tilde{G}^{(\alpha)}$ defined by

$$\tilde{G}^{(\alpha)}(x,y) = \frac{G^{(\alpha)}(x,y)}{m(x)m(y)}, \quad x, y \in M$$

satisfies the weak maximum principle with constant $b = \kappa^2$.

By Theorem 12, we then obtain the following estimate of m for each $x \in A$

$$G^{(\alpha)}_{\sigma}((Cm)^q)(x) \leq \frac{b}{q-1} Cm(x).$$

It simplifies to

$$G^{(\alpha)}_{\sigma}(m^q) \lesssim m$$

on A, and the assertion follows.

Theorem 4(Gu-Huang-S., 2023)

Let M be a complete connected non-compact Riemannian manifold. Assume that the fractional Laplacian $(-\Delta)^\alpha$ is transient. Suppose that $G^{(\alpha)}$ is quasi-metric. Then there exists a positive solution to (♠) if and only if the following two conditions hold:

$$\int_M m^q(x)d\sigma(x) < \infty, \tag{0.15}$$

and

$$\sup_{x\in M} \int_{\{y\in M:\, G^{(\alpha)}(o,y)>r^{-1}\}} G^{(\alpha)}(x,y)d\sigma(y) \lesssim r^{q-1}, \tag{0.16}$$

for all $r > a$.

Sketch of Proof of Theorem 4

" $\Longrightarrow$ " Weighted norm inequality to show that

$$\int_M (G_\sigma^{(\alpha)}(f))^s m^q d\sigma \lesssim (sb^{s-1})^s \|f\|_{L^s(\sigma)}^s.$$

holds for each $f \in L^s(\sigma)$.

Then use dulaity argument to show that

$$\|G_{m^q\sigma}^{(\alpha)}(g)\|_{L^q(\sigma)} \lesssim sb^{s-1}\|g\|_{L^q(m^q\sigma)}$$

Choose $g = \chi_A$ for some compact subset A of M with $0 < \int_A m^q d\sigma < +\infty$. By Lemma 13, we have $m \lesssim G_{m^q\sigma}^{(\alpha)}(\chi_A)$, and hence

$$\|m\|_{L^q(\sigma)} \lesssim \|G_{m^q\sigma}^{(\alpha)}(\chi_A)\|_{L^q(\sigma)} \lesssim sb^{s-1}\|\chi_A\|_{L^q(m^q\sigma)} \lesssim sb^{s-1}\left(\int_A m^q d\sigma\right)^{1/q} < +\infty.$$

first note that $r > a$ and for each $x \in M$,

$$\int_{\{y\in M:\, G^{(\alpha)}(o,y)>r^{-1}\}} G^{(\alpha)}(x,y)d\sigma(y) \le \int_{\{y\in M:\, G^{(\alpha)}(o,y)>r^{-1}\}} r^q G^{(\alpha)}(x,y)m(y)^q d\sigma(y)$$
$$\le Cr^q m(x).$$

Denote by $A_r = \{y \in M :\ G^{(\alpha)}(o,y) > r^{-1}\}$ and $\sigma_r = \chi_{A_r}\sigma$. We can rewrite the above estimate as

$$G^{(\alpha)}_{\sigma_r}1(x) \le Cr^q m(x), \quad \text{for all } x \in M.$$

Taking the q-th power of both sides and applying $G^{(\alpha)}_{\sigma_r}$, we obtain that

$$G^{(\alpha)}_{\sigma_r}((G^{(\alpha)}_{\sigma_r}1)^q) \le C^q r^{q^2} G^{(\alpha)}_{\sigma_r}(m^q) \le C^{q+1} r^{q^2} m.$$

We can then iterate this procedure to obtain for each $x \in M$ and $k \ge 1$,

$$[G^{(\alpha)}_{\sigma_r}1(x)]^{1+q+q^2+\cdots+q^k} \le b^{q+q^2+\cdots+q^k} c(q,k) C^{1+q+q^2+\cdots+q^k} r^{q^{k+1}} m,$$

It follows that by letting $k \to \infty$

$$G^{(\alpha)}_{\sigma_r}1 \le c_1 r^{q-1},$$

Proof of $\Leftarrow$

By Theorem 3, we need to show

$$m(x) \geq C G^{(\alpha)}_{\sigma}(m^q)(x).$$

First, show that

$$\int_M G^{(\alpha)}(x,y) m^q(y) d\sigma(y) < \infty,$$

and second for fix an arbitrary $x \in M$ with $G^{(\alpha)}(x,o) < 1/a$, show

$$\int_{\{y \in M : G^{(\alpha)}(x,y) \leq \kappa^2 G^{(\alpha)}(o,x)\}} G^{(\alpha)}(x,y) m^q(y) d\sigma(y) \lesssim \kappa^2 G^{(\alpha)}(o,x).$$

and

$$\int_{\{y \in M : G^{(\alpha)}(x,y) > \kappa^2 G^{(\alpha)}(o,x)\}} G^{(\alpha)}(x,y) m^q(y) d\sigma(y) \lesssim \kappa^{-q} G^{(\alpha)}(o,x).$$

Further Questions

- How to lift the regularity for fractional differential inequality?
- Parabolic analogue of the quasi-metric property? parabolic version of the potential theoretical estimates?

Many thanks for your attention!