Weyl law, Kuznecov formula, and inverse spectral problem

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Workshop on Heat Kernel and Related Topics in Hangzhou

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- N encodes the eigenvalues and their multiplicities.
- Weyl law: $N(\lambda) = C_d \mathrm{Vol}(M)\lambda^d + O(\lambda^{d-1})$.

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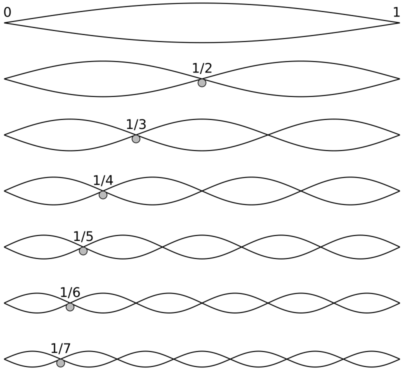

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$$\int_{-\infty}^{\infty} e^{-t\lambda^2}\, dN(\lambda) = \mathrm{tr}(e^{t\Delta}) = \frac{\mathrm{area}(\Omega)}{4\pi t} - \frac{\mathrm{length}(\partial\Omega)}{8\sqrt{\pi t}} + o(1).$$

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- We 'hear' the area of Ω and the length of $\partial\Omega$.

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- We 'hear' the area of Ω and the length of $\partial\Omega$.
- By the isoperimetric inequality, if Ω is isospectral to a disk, it must be a disk.

History

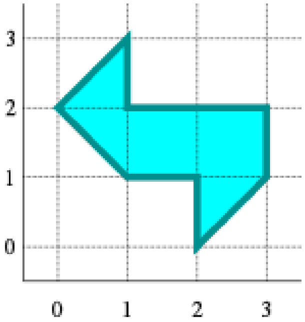
Negative results:

- [Milnor '64] Exhibits a pair of isospectral 16-dimensional tori.
- [Gordon-Webb-Wolpert '92] Exhibit a pair of isospectral polygons.
- [Buser-Conway-Doyle-Semmler '94] Generalized the method and obtained more isospectral polygons.

Positive results:

- [Kac '66] The disk is spectrally unique amongst planar domains.
- Results for various classes of drums obtained by Popov-Topalov, Vig, Hezari-Zelditch, and De Simoi-Kaloshin-Wei.
- [Hezari-Zelditch '22] Ellipses of small eccentricity are spectrally rigid.

Gordon–Webb–Wolpert counter-example



A woefully abridged history

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What do we hear when we strike a drum at a point?

- Strike the drum at $x \in M$ and let it vibrate. That is, solve

$$(\Delta - \partial_t^2)u = 0 \qquad \text{with} \qquad \begin{cases} u(0) = 0 \\ \partial_t u(0) = \delta_x. \end{cases}$$

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$$u(t, y) = \sum_j \frac{\sin(t\lambda_j)}{\lambda_j} e_j(y)\overline{e_j(x)}.$$

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- The standing wave at frequency λ is given by

$$u_\lambda(t,y) = \sum_{\lambda_j = \lambda} \frac{\sin(t\lambda_j)}{\lambda_j} e_j(y)\overline{e_j(x)}.$$

What do we hear when we strike a drum at a point?

- The perceived volume of the frequency-λ overtone is the energy

$$E(u_\lambda) = \frac{1}{2}\int_M |\nabla u_\lambda|^2 + |\partial_t u_\lambda|^2 = \frac{1}{2}\sum_{\lambda_j=\lambda} |e_j(x)|^2.$$

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- We hear the *pointwise counting function*:

Definition

The *pointwise Weyl counting function* at x is given by

$$N_x(\lambda) = \sum_{\lambda_j \leq \lambda} |e_j(x)|^2.$$

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- If $N_x(\lambda) = N_y(\lambda)$ for all λ, we say x and y are cospectral.
- If there is an isometry $M \to M$ taking $x \mapsto y$, we say x and y are similar.
- Our Question: Cospectrality $\implies$ Similarity?

Remarks on the pointwise counting function

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- Many results (e.g. Weyl law, heat trace asymptotics) about the counting function are proved by studying the pointwise counting function and integrating.
- The pointwise counting function is richly studied in its own right.

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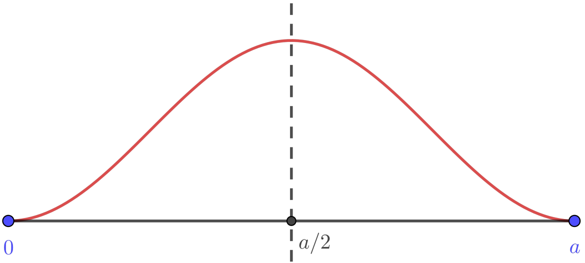

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- Both $\lambda = \pi\sqrt{1/a^2 + 1}$ and $\pi\sqrt{1/a^2 + 4}$ eigenspaces are simple.
- Both

$$\frac{4}{a} \sin^2(\pi x/a) \sin^2(\pi y) \quad \text{and} \quad \frac{4}{a} \sin^2(\pi x/a) \sin^2(2\pi y)$$

are "audible."

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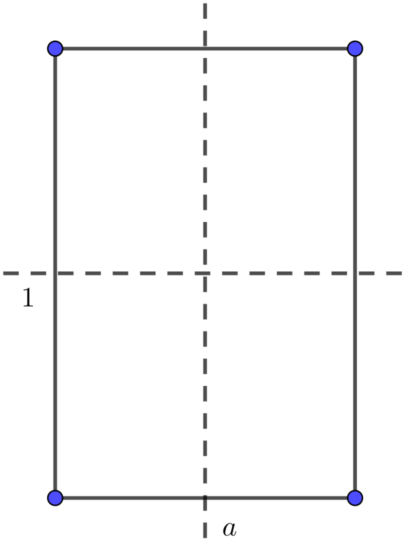

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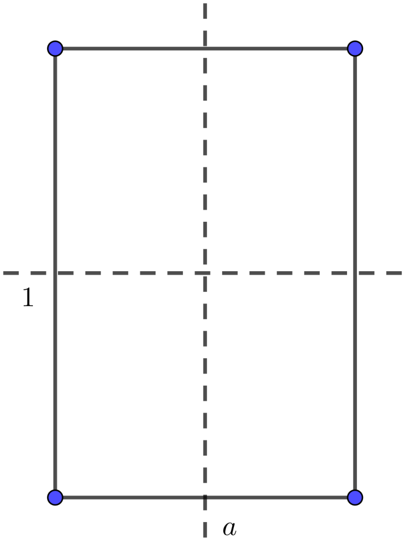

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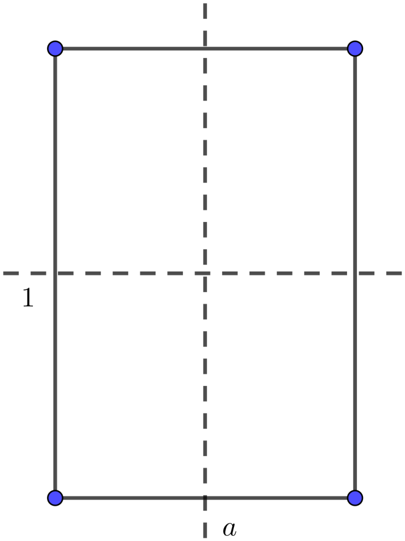

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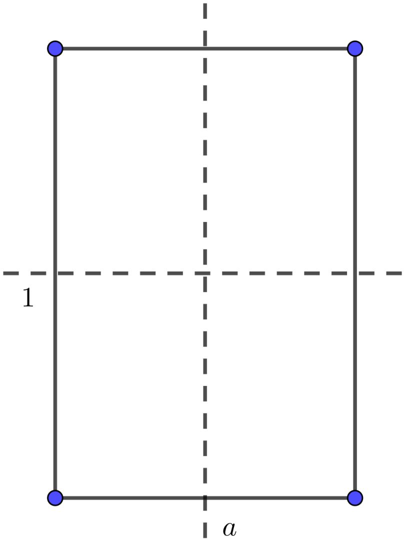

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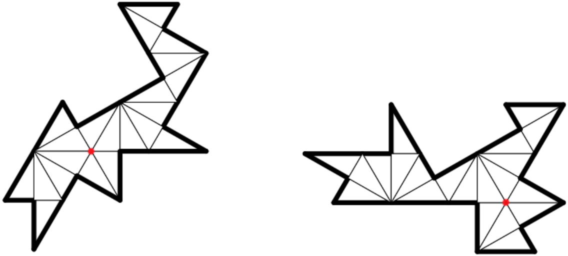

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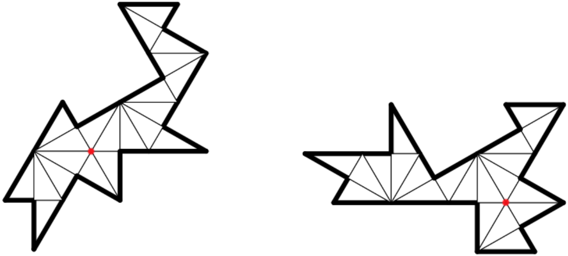

- Take M to be the disjoint union of the two domains.
- Red points are cospectral but *not* similar.

Audible quantities

Definition

Let S be any set. We say $f : M \to S$ is *audible* if it satisfies

$$f(x) = f(y) \quad \text{whenever} \quad N_x(\lambda) = N_y(\lambda) \text{ for all } \lambda.$$

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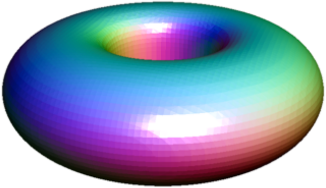
Scalar curvature $K(x)$ at x is audible since

$$\int_{-\infty}^{\infty} e^{-t\lambda^2}\, dN_x(\lambda) = e^{t\Delta_g}(x,x)$$
$$= (4\pi t)^{-n/2}\left(1 + \frac{t}{3}K(x) + O(t^2)\right).$$

Two more examples

Torus:

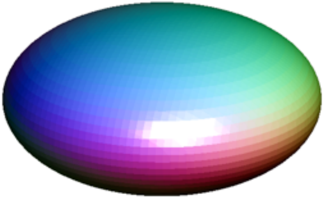
$$z^2 + (r-a)^2 = 1, \quad a > 1$$



K is an increasing function of r.

Spheroid:

$$r^2 + \frac{z^2}{a^2} = 1, \quad 0 < a \neq 1$$



K is a monotone function of $|z|$.

Summary of known and unexplored examples

We have done:

- Squares and rectangles^{D,N}
- Disks^{D,N}
- Flat Klein bottles
- Spheroids
- Tori of revolution

Trivial:

- Spheres
- Projective spheres
- Flat tori

We have not done :

- Rectangular boxes^{D,N}
- Triangles^{D,N}
- Planar ellipses^{D,N}
- Triaxial ellipsoid
- Hyperbolic surfaces

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Interpretation: Can you hear your location in a manifold?

You stand in a familiar room with your eyes closed and clap your hands once. Can you determine where you are located within the room, up to symmetry, only by listening to the resulting echos and reverberations?

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You stand in a familiar room with your eyes closed and clap your hands once. Can you determine where you are located within the room, up to symmetry, only by listening to the resulting echos and reverberations?

- Note
$$\int_{-\infty}^{\infty} \cos(t\lambda)\, dN_x(\lambda) = \cos(t\sqrt{-\Delta})(x,x) = \cos(t\sqrt{-\Delta})\delta_x(x).$$
- This is an audible distribution in t.
- N_x is uniquely determined by this distribution.
- RHS is the solution to the initial value problem
$$(\Delta - \partial_t^2)u = 0, \qquad \begin{cases} u(0) = \delta_x \\ \partial_t u(0) = 0 \end{cases}$$
evaluated at x.

Interpretation: Can you find a point by releasing Brownian particles?

You stand at an unknown point x in some planar domain and release Brownian particles that stop once they hit the boundary. You tally those particles that pass near x again for each time $t > 0$. Can you determine x up to symmetry?

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You stand at an unknown point x in some planar domain and release Brownian particles that stop once they hit the boundary. You tally those particles that pass near x again for each time $t > 0$. Can you determine x up to symmetry?

- For each $t > 0$ and x in the domain, let $p(t, x, y)$ be the probability density function for the position y of a Brownian particle at time t which started from x.
- $$p(t, x; y) = e^{\frac{1}{2}t\Delta}(x, y) = \sum_j e^{-\frac{1}{2}t\lambda_j^2} e_j(x)\overline{e_j(y)}.$$
- $$p(t, x, x) = \sum_j e^{-\frac{1}{2}t\lambda_j^2} |e_j(x)|^2 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}t\lambda^2}\, dN_x(\lambda).$$
- Extends naturally to graphs.

Interpretation: Can you find a point by releasing quantum particles?

Suppose you release a 'constrained' quantum particle at x, and you know the likelihood of its energy being λ^2 for each λ. Can you determine x up to symmetry?

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Suppose you release a 'constrained' quantum particle at x, and you know the likelihood of its energy being λ^2 for each λ. Can you determine x up to symmetry?

- $$u(t,y) = \int_M e^{it\Delta_g}(y,z)\delta_x(z)dz = \sum_{\lambda_j} e^{it\lambda_j^2}\overline{e_j(x)}e_j(y).$$

  solves the Schrödinger equation

  $$(i\partial_t - \Delta_g)u = 0 \qquad \text{with} \qquad u(0) = \delta_x.$$

- Release a 'constrained' quantum particle at x, then it becomes the superposition of various quantum states.
- Given an energy cap Λ, the probability of observing a state with energy less or equal to λ^2 is given by

  $$P_\Lambda(\lambda) = \frac{N_x(\lambda)}{N_x(\Lambda)} = \frac{\sum_{\lambda_j \le \lambda} |e_j(x)|^2}{\sum_{\lambda_j \le \Lambda} |e_j(x)|^2}.$$

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You can hear where most drums are struck

Theorem [Wyman–X '23]

Let M be a smooth, compact manifold without boundary with $\dim M \geq 2$. Then, for a residual (comeager) class of Riemannian metrics on M,

$$x = y \qquad \textit{if and only if} \qquad x,\ y \text{ are cospectral.}$$

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- If there are two cospectral points $x \neq y$, then we can make a conformal perturbation of the metric near x to disrupt the equivalence.
- We ensure there are enough such perturbations to avoid topological obstructions.

Surfaces that sound the same no matter where they are struck

Theorem [Wang–Wyman–X '23]

Suppose (M, g) is a boundary-less Riemannian surface ($\dim M = 2$) for which all points are cospectral. Then, the action of the isometry group on M is transitive.

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- Unknown in $\dim M > 2$.
- Scalar curvature is audible, and hence constant.
- Since $\dim M = 2$, (M, g) is a quotient of a space form.
- If M is a sphere, projective sphere, or flat torus, we are done.
- We eliminate the case where M is a flat Klein bottle by direct calculation.
- We eliminate the case where M is a compact hyperbolic quotient as well.

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Kuznecov Sum (Zelditch '92)

$N_H(\lambda) := \sum_{\lambda_j \leq \lambda} \left| \int_H e_j \, dV_H \right|^2 = C_{n,d} \, \mathrm{vol}^d(H) \lambda^{n-d} + O(\lambda^{n-d-1}).$

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- Possibly false in general, given the history of Kac's question.

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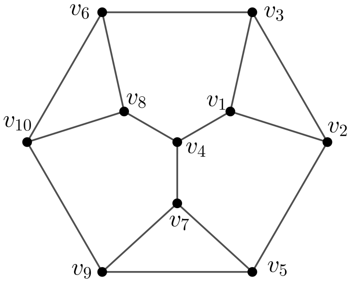

Figure: Minimal regular graph with non-similar cospectral pairs.

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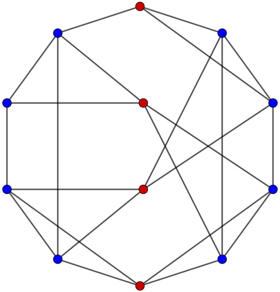

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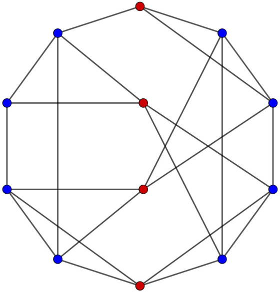

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- This graph is not planar.

Walk-regular planar graph

Theorem [Kong–Wyman–X, in preparation]

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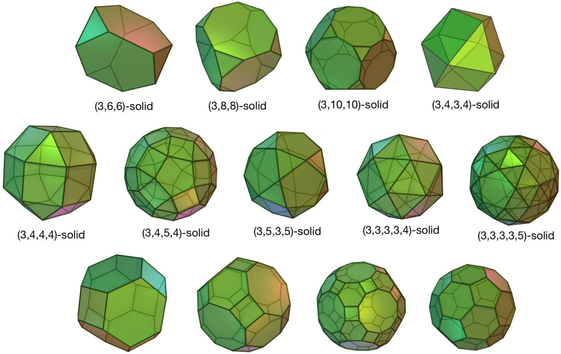
- We can classify all such graphs, non-trivial ones are the nets of Archimedean solids.



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- Conversely, are there any negative examples that are topologically connected?
- Can we say more for finite graphs?
- Are there other natural settings that we can explore?

Thank You!