On odd-normal numbers

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Heat kernel and related topics in Hangzhou, March 23, 2024

Normal number

Definition (Borel, 1909; Niven & Zuckerman, 1951)

Let $b \in \mathbb{N} \setminus \{1\}$. *A real number* $x = 0.a_1a_2\cdots = \sum_{k=1}^{\infty} a_k b^{-k} \in [0,1]$ *is normal to base* b *(or* b*-normal for short) if, and only if,* $\forall k \in \mathbb{N}$ *and any combination* $D_k = (d_1, \ldots, d_k)$ *of* k *digits,* $d_i \in \{0, 1, \ldots, b-1\}$,

$$\lim_{N\to\infty} \frac{1}{N} \sharp \left\{0 \le j \le N-1 : (a_{j+1}, \ldots, a_{j+k}) = (d_1, \ldots, d_k)\right\} = \frac{1}{b^k}.$$

.

- $x = 0.2113461137732311 3\cdots$, $D_3 = (1, 1, 3)$, $b = 10$.

Normal number

- É. Borel, Les probabilités dénombrables et leurs applications arithmétiques, Rend. Circ. Math. Palermo 27 (1909), 247-271.

- I. Niven $\&$ H. S. Zuckerman, On the definition of normal numbers, Pacific J. Math. 1 (1951), 103-109.

1) Rational numbers can not be normal to any base:

$$\frac{1}{7} = 0.142857142857\cdots \quad \text{(10-ary expansion)},$$

where only the combination of $D_6 = (1, 4, 2, 8, 5, 7)$ occurs infinitely often.

Normal number

2) The explicit example of a real number normal to a given base:

$$\xi_c := 0.1234567891011121314\cdots$$

(the sequence of all positive integers), is normal to base 10.

- D. G. Champernowne, The construction of decimals normal in the scale of ten, J. London Math. Soc. 8 (1933), 254 – 260.

Normal number

3) It is not known whether or not π, e, $\sqrt{2}$, $\dots$ are normal to a fixed base.

Theorem 1 (Borel, 1909)

Almost every (Lebesgue measure) $x \in [0,1]$ is normal to all $b \geq 2$.

- Generally, it is difficult to find normal or non-normal numbers.
  - Y. Bugeaud, Distribution modulo one and Diophantine approximation, Cambridge Tracts in Mathematics, 193. Cambridge University Press, Cambridge, 2012.

Normality & uniform distribution

Theorem 2 (Wall, 1949)

Let $b \in \mathbb{N} \setminus \{1\}$. Then $x \in [0,1]$ is normal to base b if, and only if, the sequence $(b^n x)_{n \geq 0}$ is uniformly distributed modulo one.

- D. D. Wall, Normal numbers. PhD thesis, University of California, Berkeley, CA, 1949.

- A sequence $(x_n)_{n \geq 1}$ of real numbers is said to be uniformly distributed modulo one if, for any α, β with $0 \leq \alpha < \beta \leq 1$, the fractional part of x_n, denoted by $\{x_n\}$, satisfies

$$\lim_{N \to \infty} \frac{1}{N} \sharp \{1 \leq n \leq N : \ \{x_n\} \in [\alpha, \beta)\} = \beta - \alpha.$$

Normality & uniform distribution

Theorem 3 (Weyl, 1916)

$x \in [0,1]$ is normal to base $b \in \mathbb{N} \setminus \{1\}$ if, and only if, for any $h \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i h b^n x} = 0 \quad \textit{(Weyl's criterion).}$$

- H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins., Math. Ann. 77 (1916), 313-352.

Theorem 4 (Davenport-Erdős-LeVeque, 1963)

Let μ be a finite Borel measure on $[0,1]$, and $b \in \mathbb{N} \setminus \{1\}$. If, for any $h \in \mathbb{Z} \setminus \{0\}$,

$$\sum_{N=1}^{\infty} \frac{1}{N} \int_0^1 \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i h b^n x} \right|^2 \mathrm{d}\mu(x) < \infty,$$

$$\iff \sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m,n=0}^{N-1} \left| \widehat{\mu}\left(h(b^n - b^m)\right) \right| < \infty$$

then, for μ-a.e. $x \in [0,1]$ is normal to base b.

Measure & normality

Example

Let $C_{1/3}$ *be the middle* $1/3$ *Cantor set, i.e.*

$$C_{1/3} = \left\{ x \in [0,1] : \; x = \sum_{j=1}^{\infty} \varepsilon_j 3^{-j}, \quad \varepsilon_j \in \{0,2\} \right\}.$$

Any $x \in C_{1/3}$ *is not* 3*-normal and* $C_{1/3}$ *can not support any Rajchman measure.*

Rajchman measure

Definition (Rajchman measure)

Let μ be a finite Borel measure on $[0,1]$. If $\widehat{\mu}(n) \to \infty$ as $|n| \to \infty$, μ is called a Rajchman measure, where

$$\widehat{\mu}(n) := \int_0^1 e^{-2\pi i n x}\, \mathrm{d}\mu(x).$$

Question (Kahane & Salem, 1964)

Can a set of non-normal numbers support a Rajchman measure?

- J. P. Kahane and R. Salem, Distribution modulo 1 and sets of uniqueness, Bull. Amer. Math. Soc. 70 (1964), 259-261.

Set of uniqueness & multiplicity

Definition

A set $E \subset [0,1]$ is called a set of uniqueness, if any trigonometric series of the form

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x} = 0 \ \ \forall\, x \in [0,1] \setminus E \Longrightarrow c_n \equiv 0.$$

Otherwise, it is called a set of multiplicity.
Let $\mathcal{U}$ denote the set of all sets of uniqueness.

- Originated from 1869, Riemann, Heine, Cantor .... It is well known that

$$\text{countable set} \subset \mathcal{U} \cap \text{L. measurable} \subset \text{L. null}.$$

Set of uniqueness & multiplicity

Theorem 5 (Piatetcki-Shapiro, 1952; Kahane & Salem, 1963)

Let $E \subset [0,1]$ *be a closed set and* μ *be a probability Borel measure on* $[0,1]$ *with* $\mu(E) = 1$. *Then the following are equivalent:*
(i) μ *is a Rajchman measure;*
(ii) E *is a set of multiplicity.*

Rajchman measure

- Back to Kahane & Salem's question, it was answered in affirmative by Lyons for the set of numbers which are not 2-normal.
  - R. Lyons, The measure of nonnormal sets, Invent. Math. 83 (1986), 605-616.
- The main objective of this talk is to study the further properties of Lyons's measure.

Lyons's measure

- Let $(K_i)_{i=1}^{\infty} \subset \mathbb{N}$ with $K_i > K_{i-1}$, $K_0 = 0$, and
$$\liminf_{i\to\infty} \frac{K_i}{K_{i-1}} > 1.$$
Suppose that $(\varepsilon_i)_{i=1}^{\infty} \subset [0,1]$ satisfying $\sum_i \varepsilon_i = \infty$. Define
$$\mu := \mathop{\ast}_{i=1}^{\infty} \left\{ \varepsilon_i \delta(0) + (1-\varepsilon_i) \mathop{\ast}_{k=K_{i-1}+1}^{K_i} \left[\frac{1}{2}\delta(0) + \frac{1}{2}\delta(2^{-k}) \right] \right\}.$$
- μ is a probability measure on $[0,1]$. If $\varepsilon_i \equiv 0$, then μ is Lebesgue measure on $[0,1]$.

- For $2^{K_{i-1}-1} \leq n < 2^{K_i-1}$,

$$|\widehat{\mu}(n)| \leq \varepsilon_i\varepsilon_{i-1} + \varepsilon_i + \varepsilon_{i-1} + 2^{-(K_i-K_{i-2})}\left|\frac{1-e^{-2\pi in2^{-K_{i-2}}}}{1-e^{-2\pi in2^{-K_i}}}\right|$$
$$\leq \varepsilon_i\varepsilon_{i-1} + \varepsilon_i + \varepsilon_{i-1} + 2^{-(K_{i-1}-K_{i-2})}.$$

Lyons's measure

- (Borel-Cantelli lemma) Let $(\Omega, \mathcal{F}, P)$ be a probability space. If the events $\{A_i\}_{i\in\mathbb{N}}$ are independent, then $\sum_{i=1}^{\infty} P(A_i) = \infty$ implies $P(\limsup_{i\to\infty} A_i) = 1$.
- For any $x \in [0,1]$, let $x = \sum_{k=1}^{\infty} d_k(x)2^{-k}$ be the binary expansion of x. Then

$$A_i := \{x \in [0,1] : \ d_k(x) = 0 \text{ for all } K_{i-1}+1 \le k \le K_i\}$$

and

$$\mu(A_i) = \varepsilon_i + (1-\varepsilon_i)2^{K_{i-1}-K_i}.$$

Main result

Let $\mathcal{O}$ and $\mathcal{E}$ denote respectively the set of all odd and even integers in $\mathbb{N} \setminus \{1\}$.

Theorem 6 (Pramanik-Z.,2024)

Let μ be the probability measure on $[0,1]$ constructed by Lyons, with $K_i := K^{i\lfloor\sqrt{\log i}\rfloor}$ and $\varepsilon_i := i^{-1}$ for some integer $K \geq 10$. Then for any $\kappa > 0$, there exists $C_\kappa > 0$ such that

$$|\widehat{\mu}(n)| \leq (\log\log|n|)^{-1+\kappa} \ \ \forall n \in \mathbb{Z} \text{ with } |n| \geq C_\kappa. \tag{1}$$

Moreover, μ-almost every $x \in [0,1]$ is not normal to any $b \in \mathcal{E}$, but normal to any $b \in \mathcal{O}$.

Main result

- The decay rate in (1) is sharp, up to κ-loss, in the following sense: It is known that if a set E supports a measure ν such that for some $\kappa > 0$, $|\widehat{\nu}(n)| \le C_\kappa(\log\log|n|)^{-1-\kappa}$, then ν-a.e. $x \in E$ is normal to all $b \ge 2$.

  - A. Pollington, S. Velani, A. Zafeiropoulos, E. Zorin, Inhomogeneous Diophantine approximation on M_0-sets with restricted denominators, Int. Math. Res. Not. (2022), 8571-8643.

Proof overview

- Step 1: show that μ-almost every x is not normal to any $b \in \mathcal{E}$: Weyl's uniform distribution criterion.
- Step 2: show μ-almost every x is normal to every $b \in \mathcal{O}$: verifying Davenport-Erdös-LeVeque's theorem by using number-theoretic result of Schmidt.

Step 1

- $\mathcal{B}_i := \{K_{i-1}+1, K_{i-1}+2, \dots, K_i\}$,
  $A_i := \{x \in [0,1) : \ d_k(x) = 0 \text{ for all } k \in \mathcal{B}_i\}$,

  $$A := \limsup_{i\to\infty} A_i = \bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_i$$

  $$= \{x \in [0,1] : \ d_k(x) = 0 \text{ for all } k \in \mathcal{B}_i \text{ for infinitely many } i\}.$$

  $\mu(A) = 1$ and for all $x \in A$,

  $$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \text{Re}\left(e^{-2\pi i b^k x}\right) = 1, \quad \forall b \in 2\mathbb{N}.$$

Step 2

- Applying Theorem of DEL, it aims to establish, for any odd integer $b \geq 2$,

$$\sum_{N=1}^{\infty} N^{-3} \sum_{u=1}^{N} \sum_{v=1}^{N} \hat{\mu}(hb^u(b^v - 1)) < \infty.$$

- For any $\eta := \sum_{k=0}^{\infty} d_k(\eta) 2^k \in \mathbb{Z}$,

$$\hat{\mu}(\eta) = \prod_{i=1}^{\infty} \left[\varepsilon_i + (1 - \varepsilon_i)\mathcal{C}_i(\eta)\right].$$

Step 2

- $$\mathcal{C}_i(\eta) := \prod_{k=K_{i-1}+1}^{K_i} \left[\frac{1}{2}\left(1+e^{-2\pi i 2^{-k}\eta}\right)\right]$$

  and

  $$\frac{1}{2}\left(1+e^{-2\pi i 2^{-k}\eta}\right) = e^{-\pi i 2^{-k}\eta}\cos(\pi 2^{-k}\eta).$$

- If $\boldsymbol{\eta}_i := \{d_k(\eta) : \ K_{i-1}+1 \leq k \leq K_i\}$ are all 0 or 1, then $\mathcal{C}_i \sim 1$ (independent of l). Long unbroken runs of single digits in $\boldsymbol{\eta}_i$ lead to relatively large value of $\mathcal{C}_i$.

Step 2

- This suggests that for $\mathcal{C}_i$ to attain values that are significantly smaller than its maximum, consecutive digits in $\boldsymbol{\eta}_i$ must change, and change frequently.
- In the binary expansion of

$$\eta := hb^u(b^v - 1) = \sum_{k=0}^{\infty} d_k 2^k,$$

for most of $(u, v) \in \{1, \ldots, N\}$, the digits $(d_k)_k$ change frequently.

Step 2

Lemma 7 (Schmidt, 1960)

Given any odd integer $r \in \mathbb{N} \setminus \{1\}$*, there exists a constant* $c(r) > 0$ *as follows. For any integer* $\rho \in \mathbb{Z} \setminus \{0\}$*, write* $\rho := \rho_2 \rho_2'$ *with* $\rho_2 = 2^m$ *for some* $m \in \mathbb{N} \cup \{0\}$ *and* $2 \nmid \rho_2'$*. Then for every* $k \geq 1$ *and every* $\sigma \in \{0, 1, \ldots, 2^k - 1\}$*,*

$$\sharp \left\{ m \in \{0, 1, \ldots, 2^k - 1\} : \ \rho r^m = \sigma \mod 2^k \right\} \leq c(r) \rho_2.$$

- W. M. Schmidt, On normal numbers, Pacific J. Math. 10 (1960), 661 – 672.

Step 2

- For any $k \in \mathbb{N}$ and $n \in \{0, 1, \ldots, 3^k - 1\}$, let

  $$r^n = \lambda_0(n)3^0 + \lambda_1(n)3^1 + \cdots + \lambda_i(n)3^i + \cdots, \quad \lambda_i(n) \in \{0, 1, 2\},$$

  be the 3-ary expansion of r^n.
  Then $(\lambda_l(n), \lambda_{l+1}(n), \ldots, \lambda_{l+k-1}(n))$ take precisely once every one of the 3^k possible values of $\{0, 1, 2\}^k$, as n runs from 0 to $3^k - 1$, where $l \in \mathbb{N} \cup \{0\}$ is such that $r \equiv 1 \pmod{3^l}$ and $r \not\equiv 1 \pmod{3^{l+1}}$.

- We say that $r, s \in \mathbb{N} \setminus \{1\}$ are multiplicatively dependent, and write

$$r \sim s \quad \text{if} \quad \frac{\log r}{\log s} \in \mathbb{Q}, \ \ i.e., \ \ \text{if} \ \exists\, m, n \in \mathbb{N}, \ s.t. \ r^m = s^n.$$

Otherwise, we put $r \not\sim s$ and call r, s multiplicatively independent.

Theorem 8 (Schmidt, 1960)

(i) *Let* $r, s \in \mathbb{N} \setminus \{1\}$. *If* $r \sim s$, *any* $x \in [0,1]$ *normal to base* r *is normal to base* s.
(ii) *For any* $x \in C_{1/3}$, x *is normal to any base* $b \not\sim 3$.

- J. W. S. Cassels, On a problem of Steinhaus about normal numbers, Colloq. Math. 7 (1959), 95 – 101.
- W. M. Schmidt, On normal numbers, Pacific J. Math. 10 (1960), 661 – 672.

Normality & non-normality to restricted bases

- For $\mathcal{B}, \mathcal{B}' \subset \mathbb{N} \setminus \{1\}$, $\mathcal{B}$ and $\mathcal{B}'$ are called compatible if $\forall (b, b') \in \mathcal{B} \times \mathcal{B}'$, $b \nsim b'$.
- For any compatible $\mathcal{B}, \mathcal{B}'$, define

$$\mathcal{N}(\mathcal{B}, \mathcal{B}') := \{x \in [0,1] : \ x \text{ is normal to } \forall b \in \mathcal{B}, \\ \text{but not normal to } \forall b' \in \mathcal{B}'\}.$$

- For any compatible $\mathcal{B} \cup \mathcal{B}' = \mathbb{N} \setminus \{1\}$, Schmidt showed that $\mathcal{N}(\mathcal{B}, \mathcal{B}') \neq \emptyset$, and Pollington further showed that

$$\dim_{\mathrm{H}} \mathcal{N}(\mathcal{B}, \mathcal{B}') = 1.$$

Normality & non-normality to restricted bases

- A. D. Pollington, The Hausdorff dimension of a set of normal numbers, Pacific J. Math. 95 (1981), 193-204.
- W. M. Schmidt, Über die Normalität von Zahlen zu verschiedenen Basen, Acta Arith. 7 (1961/62), 299 – 309.

- For different proof of the result

$$\mathcal{N}(\mathcal{B}, \mathcal{B}') \neq \emptyset, \quad \text{and} \quad \dim_{\mathrm{H}} \mathcal{N}(\mathcal{B}, \mathcal{B}') = 1,$$

see also the work of Moran et al., using the Riesz product measure.

- G. Brown, W. Moran & C. E. M. Pearce, Riesz products and normal numbers, J. London Math. Soc. (2) 32 (1985), 12-18.

- G. Brown, W. Moran & C. E. M. Pearce, Riesz products, Hausdorff dimension and normal numbers, Math. Proc. Camb. Phil. Soc. (1987), 529-540.

- W. Moran & A. D. Pollington, Metrical results on normality to distinct bases, Journal of number theory (1995), 180-189.

- The set $\mathcal{N}(\mathcal{B}, \mathcal{B}')$ constructed by Schmidt and Pollington is an H-set, i.e. $\exists$ some nonempty open interval $I \subset [0,1]$ and $0 \leq n_0 < n_1 < n_2 < \cdots$, s.t. for any $k \in \mathbb{N}$, $(n_k \mathcal{N}) \cap I = \emptyset$.
- Rajchman showed that every H-set is a set of uniqueness, which can not support a Rajchman measure.

Normality & non-normality to restricted bases

Question

For any compatible $\mathcal{B}, \mathcal{B}'$ such that $\mathcal{B} \cup \mathcal{B}' = \mathbb{N} \setminus \{1\}$, whether $\mathcal{N}(\mathcal{B}, \mathcal{B}')$ can support a Rajchman measure?

Thank you!